7.5 Rates of Change and Motion in a Line

We have already learned that a derivative is the slope of a line, but what's the difference between average and instantaneous rates of change?

Definition: The **average rate of change** is slope of the *secant* line, and thus, simply the slope formula you learned in Algebra, $\frac{\text{change in y}}{\text{change in x}}$.

Definition: The **instantaneous rate of change** is the slope of the *tangent* line, thus, the derivative at a certain instance.

Example 16: A diver jumps from a platform at t = 0 seconds. The distance of the diver above water level at time *t* is given by $s(t) = -4.9t^2 + 4.9t + 10$, where s is in meters.

a.) Find the average velocity of the diver over the given time intervals.

i [1,2]	ii [1.5, 1]	iii [1.1,1]	iv[1.01,1]
$s(1)=10$ $s(2)=.2$ $V_{avg}=\frac{10-.2}{1-2}=$ $=\frac{9.8}{-1}$ $=-9.8$ m/s	$s(1.5)=6.325$ $s(1)=10$ $V_{avg}=\frac{10-6.325}{1-1.5}$ $=-7.35$ m/s	$s(1.1)=9.461$ $s(1)=10$ $V_{avg}=\frac{10-9.461}{1-1.1}$ $=-5.39$ m/s	$s(1.01)=9.95051$ $s(1)=10$ $V_{avg}=\frac{10-9.95051}{1-1.01}$ $=-4.949$ m/s

b.) Find the instantaneous velocity of the diver at t = 1 second.

→derivative $s'(t)=-9.8t+4.9$

$s'(1) = -9.8+4.9 = -4.9$ m/s

*Food for thought? What do you notice about part a, compared with part b? → the smaller the interval the closer you get to the actual answer

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Example 17: During one month, the temperature of the water in a pond is modeled by the function $C(t) = 20 + 9te^{-t/3}$, where t is measured in days and C is measured in degrees Celcius.

a. Find the average rate of change in temperature in the first 15 days of the month.

$\frac{C(0)-C(15)}{0-15} = \frac{20-[20+135e^{-5}]}{-15} = \frac{-135e^{-5}}{-15} = 9e^{-5} = \frac{9}{e^5} \approx .0606$ °C per day

b. Find the rate of change in temperature on day 15. $C(t) = 20 + 9t\,e^{-\frac{1}{3}t}$ Product + Chain rule!

$C'(t) = 9t\,e^{-\frac{1}{3}t} \cdot -\frac{1}{3} + 9 \cdot e^{-\frac{1}{3}t}$

$-3te^{-t/3} + 9e^{-t/3}$

$t=15$ → -0.243°C a day

Definition: If an object is moving along a straight line, its position from an origin at any time *t* can be modeled by s(t), called the **displacement function**. (In Calculus, we called this the position function).

Definition: The **initial position** is the position when t = 0, hence, s(0).

As previously stated, the instantaneous rate of change of displacement is the **velocity function.**
v(t) = s '(t).
When v(t) > 0, the object is moving to the right (or up).
When v(t) < 0, the object is moving to the left (or down).
When v(t) = 0, the object is at rest.

*__Speed__ is the absolute value of velocity.

Example 18: A particle moves in a straight line with a displacement of s meters t seconds after leaving a fixed point. The displacement function is given by $s(t) = 2t^3 - 21t^2 + 60t + 3$, for $t \geq 0$.

a.) Find the velocity of the particle at any time t.

$v(t) = s'(t) = 6t^2 - 42t + 60$

b.) Find the initial position and initial velocity of the particle.

$s(0) = 3m$ $\qquad v(0) = s'(0) = 60\ m/s$

c.) Find when the particle is at rest.

$v(t) = 0 \quad 6t^2 - 42t + 60 = 0 \rightarrow t^2 - 7t + 10 = 0$
$(t-5)(t-2) = 0$
$t = 2$ and 5 seconds

d.) Find when the particle is moving left and when the particle is moving right.

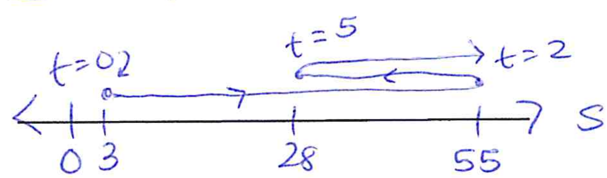
plug in values to v(t) on both sides of 2 and 5

+ | − | +
2 | 5

right (0,2) (5,∞)
left (2,5)

e.) Draw a motion diagram for the particle.

$S(2) = 55m$
$S(5) = 28m$



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As previously stated, the instantaneous rate of change of displacement is the **acceleration function.** $a(t) = s''(t)$.
When $a(t) > 0$, the velocity of the object is increasing.
When $a(t) < 0$, the velocity of the object is decreasing.
When $a(t) = 0$, the velocity is constant.

When velocity and acceleration have the same sign, the object in motion is speeding up
When velocity and acceleration have different signs, the object in motion is slowing down

Example 19: For the displacement function from Example 18, $s(t) = 2t^3 - 21t^2 + 60t + 3$, with s in meters and $t \geq 0$ seconds, we found that $v(t) = 6t^2 - 42t + 60$.

a.) Find the average acceleration of the particle from t = 1 second to t = 4 seconds.

$$a_{average} = \frac{v(1) - v(4)}{1-4} = \frac{24-(-12)}{-3} = -12 \ m/s^2$$

or $-12 \ ms^{-2}$

b.) Find the instantaneous acceleration of the particle at t = 3 seconds. Explain the meaning of your answer.

$$a(t) = v'(t) = 12t - 42$$

$$a(3) = v'(3) = 12(3) - 42 = -6 \ m/s^2$$

Explain: velocity is decreasing 6 m/s each second at $t = 3$

Example 20: For the displacement function from Example 18, $s(t) = 2t^3 - 21t^2 + 60t + 3$, with s in meters and $t \geq 0$ seconds, we found that $v(t) = 6t^2 - 42t + 60$ and $a(t) = 12t - 42$.

a.) Find the speed of the particle at t = 3 seconds and determine whether the particle is speeding up or slowing down when t = 3 seconds.

Speed $= |v(t)| = |v(3)| = |-12| = 12 \ m/s$

Since $a(3) = -6$

The particle is speeding up

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b.) During $0 \leq t \leq 10$ seconds, find the intervals when the particle is speeding up and when it is slowing down.

$a(t) = 12t - 42 = 0$

$t = 3.5$

$v(t) = 0$

$t = 2 + 5$

$v(t)$: + on (0, 2), − on (2, 5), + on (5, 10) (from example 18)

0 2 3.5 5 10

$a(t)$: − on (0, 3.5), + on (3.5, 10)

0 2 3.5 5 10

speed up $(2, 3.5) \cup (5, 10)$
(where signs are the same)

Slow down $(0, 2) \cup (3.5, 5)$
(where signs are different)