7.1 Limits and Convergence

Definitions: The term **convergent** is used for sequences where the term number in the sequence increases and approaches a fixed number known as a **limit.** All other sequences are **divergent.**

Notation: $\lim_{n\to\infty} u_n = L$ Read as "The limit as n approaches infinity of u_n equals L.

Example 1: Determine whether each sequence is convergent or divergent. If the sequence is convergent, give the limit of the sequence.

a.) 0.3, 0.33, 0.333, 0.3333, … convergent $\lim_{x\to\infty} f(x) = 1/3$

b.) 2, 4, 8, 16, … divergent

c.) $\frac{1}{5}, \frac{6}{25}, \frac{31}{125}, \frac{156}{625}, \frac{781}{3125}, \ldots$ convergent $\lim_{x\to\infty} f(x) = 1/4$

d.) 1, -1, 1, -1, … divergent

Thrmk A

What do limits mean?

$\lim_{x\to c} f(x) = L$ means that the function f(x) is headed towards L when the x-values get closer and closer to x = c.

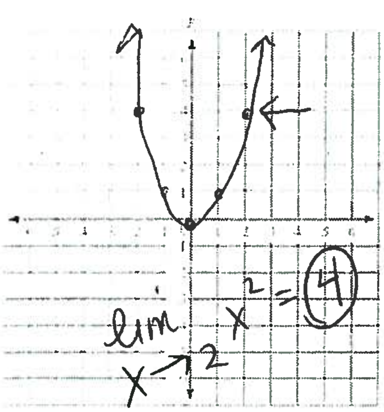
Remember, a limit is where function's headed, NOT where the function's at.

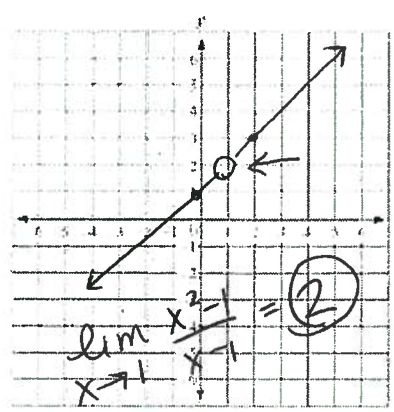
****Remember, a limit only exists when the limit from the left = limit from the right****

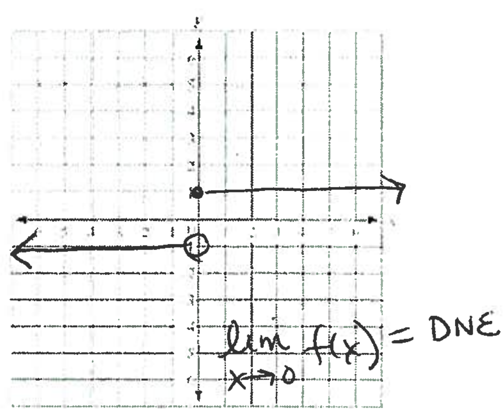
Example 2: Using a GCD, examine each function graphically and numerically. Find the limit or state that it doesn't exist.

a.) $\lim_{x\to 2} x^2$ b.) $\lim_{x\to 1} \frac{x^2-1}{x-1}$ ↗ $\frac{(x+1)(x-1)}{x-1}$ c.) $\lim_{x\to 0} f(x);$ *where* $f(x) = \begin{cases} 1 \text{ for } x \geq 0 \\ -1 \text{ for } x < 0 \end{cases}$

Thmk B







* approaches 2 from both sides

* approaches -1 from left and 1 from right MUST be same on both sides

What about using algebra?

ⓐ $2^2 = 4$

ⓑ $\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{(x-1)} = x+1$

$1+1 = 2$

Finding Limits Algebraically Involving Infinity

Limits to ∞ or -∞ can be found by plugging in very large or very small (very large negatives) numbers to see the value your function approaches.

Example: Use a calculator to estimate $\lim\limits_{x\to\infty}\left[\sqrt{x^2+1}-(x+1)\right]$

Since my limit is approaching ∞, plug in large values
(for -∞ use small values… which would be large negative values)

x	100	1000	10,000	100,000
y	-0.995000125	-.9995000001	-.99995	-.999995

My limit appears to be -1.

You can use an algebra "shortcut" to evaluate at infinity as well...

Finally, we are interested also in problems of the type: $\lim\limits_{x\to\pm\infty} f(x)$. Here are the rules:

Write $f(x)$ as a fraction. 1) If the highest power of x appears in the denominator (bottom heavy), $\lim\limits_{x\to\pm\infty} f(x)=0$

2) If the highest power of x appears in the numerator (top heavy), $\lim\limits_{x\to\pm\infty} f(x)=\pm\infty$
plug in very large or small numbers and determine the sign of the answer

3) If the highest power of x appears both in the numerator and denominator (powers equal), $\lim\limits_{x\to\pm\infty} f(x)=\dfrac{\text{coefficient of numerator's highest power}}{\text{coefficient of denominator's highest power}}$

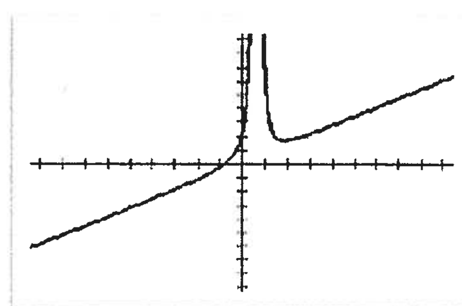
Example 1:

$$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{4x^3-4x^2+5}{6x^2-7x+2}$$

Plot1 Plot2 Plot3
$\backslash Y_1 = (4X^3-4X^2+5)/(6X^2-7X+2)$

Using the shortcut above, the higher power of x is on the top so the limit is ∞ or $-\infty$

Use the graph or a table of values to evaluate the limit at ∞ to tell for sure.



Look at the far right hand side or plug in a BIG positive number (since the limit is to ∞)

Or f(10,000) = 6666.8

In this case, since the right is headed up and f(∞) is a big positive number, the limit is ∞.

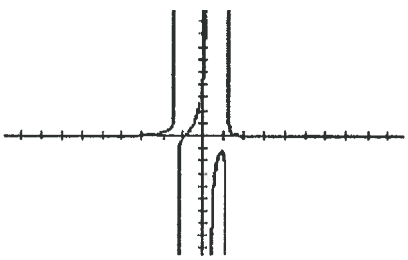
Example 2:

$$\lim_{x\to+\infty} f(x) = \frac{2x^3 - 4x^2 + 5}{6x^5 - 7x^3 + 2x^2 - 4x + 1}$$

In this case since the higher power of x is on the bottom, the limit is 0.

You can verify via graph and algebra.

In this case look at the right hand side of the graph or plug in a big positive number since the limit is to ∞.



Or f(10,000) = 3.3E-9 which means 3.3×10^{-9} or essentially 0.

Example 3:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{4x^3 - 3x^2 + 2}{7x^3 + 5x^2 - x + 6}$$

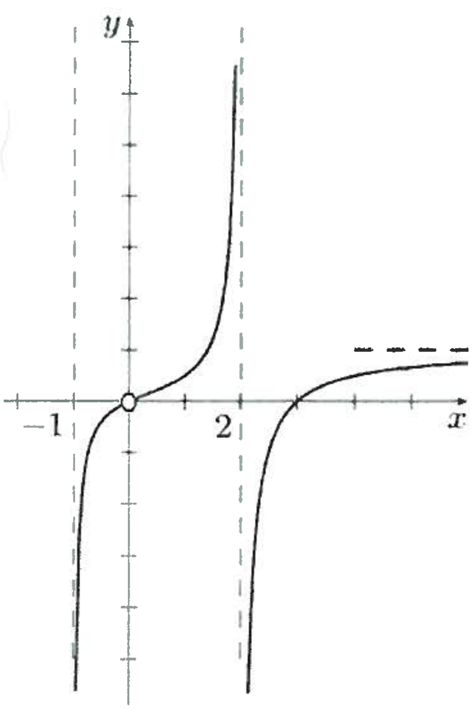
In this case since the powers of x are the same on the top and bottom, the limit is the ratio of the leading coefficients.

So the limit is 4/7

You can verify via graph and algebra.

In this case look at the left hand side of the graph or plug in a big negative number since the limit is to $-\infty$. Since the limit is a yucky number, you will only get an estimate using these methods.

Limit Practice...



Use the graph of f(x) at the left to answer the questions below.

a) f(1) = 0.5 (approx.)

b) f(3) = 0

c) f(0) = DNE

d) $\lim_{x\to 0^-} f(x) = 0$

e) $\lim_{x\to 0^+} f(x) = 0$

f) $\lim_{x\to 0} f(x) = 0$

g) $\lim_{x\to 2^-} f(x) = \infty$

h) $\lim_{x\to 2^+} f(x) = -\infty$

i) $\lim_{x\to 2} f(x) = \text{DNE}$

j) $\lim_{x\to -\infty} f(x) = -\infty$

k) $\lim_{x\to \infty} f(x) = 1$

worksheet