

Section 12.5 χ^2 Test of Independence ~ Manual Working

In the last section we will let the calculator do all the work for us. In this section we will do some of the work by hand. On the exam, most of the work will be done on the graphing calculator. If you choose to do chi-square in your Internal Assessment then you should do as much as you can by hand.

The table to the right shows the results of a sample of 400 randomly selected adults classified according to gender and regular exercise.

	Regular exercise	No regular exercise	Sum
Male	110	106	216
Female	98	86	184
Sum	208	192	400

Notice the table has sums for each column and row. We need that information to do some of this by hand.

To test whether or not gender and regular exercise are independent, we first consider only the sum values of the contingency table. We then calculate the values we would expect to obtain if the variables were independent.

	Regular exercise	No regular exercise	Sum
Male			216
Female			184
Sum	208	192	400

For example, if gender and regular exercise were independent, then

$$P(\text{male} \cap \text{regular exercise}) = P(\text{male}) \times P(\text{exercise}).$$

$$P(\text{male} \cap \text{regular exercise}) = P(\text{male}) \times P(\text{exercise}) = \frac{216}{400} \times \frac{208}{400}$$

So, in this sample of 400 adults, we would expect

$$400 \times \left(\frac{216}{400} \times \frac{208}{400} \right) = \frac{216 \times 208}{400} = 112.32$$

112.32 people to be male and exercise regularly.

We can perform similar calculations for each cell to complete an expected frequency table. This displays the values we would expect to obtain if the variables were independent.

	Regular exercise	No regular exercise	Sum
Male	$\frac{216 \times 208}{400} = 112.32$	$\frac{216 \times 192}{400} = 103.68$	216
Female	$\frac{184 \times 208}{400} = 95.68$	$\frac{184 \times 192}{400} = 88.32$	184
Sum	208	192	400

On the exam you need to be able to find the expected frequency for a requested cell... for example male and regular exercise. You will NOT have to do an entire table!

The χ^2 test examines the difference between the observed values we obtained from our sample, and the expected values we have calculated.

$$\chi^2_{calc} = \sum \frac{(f_o - f_e)^2}{f_e}$$

where f_o is an observed frequency (given in the original table) and f_e is an expected frequency (computed using the sums).

If the variables are independent, the observed and expected values will be very similar. This means that the values of $(f_o - f_e)$ will be small, and therefore χ^2_{calc} will also be small.

If the variables are not independent, the observed values will differ significantly from the expected values. The values of $(f_o - f_e)$ will be large, and therefore χ^2_{calc} will also be large.

Luckily, we can make the calculator find χ^2_{calc} for us! That is enough for the exam. But as previously stated, if this is done in your internal assessment, you should compute it by hand.

On the exam you could expect to be asked to find the degrees of freedom and the expected values for your table by hand.

Reminder... For a contingency table which r rows and c columns, degrees of freedom = $(r - 1)(c - 1)$. The row and column numbers do not include the sums.

Example 1. Complete the expected frequency table that corresponds to the below contingency table describing the favorite color of t-shirt for men and women. Then find the value of χ^2 .

	Black	White	Red	Blue
Male	48	12	33	57
Female	35	46	42	27

	Black	White	Red	Blue	Sum
Male					150
Female					150
Sum	83	58	75	84	300

Expected Frequencies...

$$\begin{aligned} \text{Male} \cap \text{Black} &= 300 \times \left(\frac{150}{300} \times \frac{83}{300} \right) = \frac{150 \times 83}{300} \\ &= 41.5 \end{aligned}$$

$$\begin{aligned} \text{Female} \cap \text{Black} &= 300 \times \left(\frac{150}{300} \times \frac{83}{300} \right) = \frac{150 \times 83}{300} \\ &= 41.5 \end{aligned}$$

$$\begin{aligned} \text{Male} \cap \text{White} &= 300 \times \left(\frac{150}{300} \times \frac{58}{300} \right) = \frac{150 \times 58}{300} \\ &= 29 \end{aligned}$$

$$\begin{aligned} \text{Female} \cap \text{White} &= 300 \times \left(\frac{150}{300} \times \frac{58}{300} \right) = \frac{150 \times 58}{300} \\ &= 29 \end{aligned}$$

$$\begin{aligned} \text{Male} \cap \text{Red} &= 300 \times \left(\frac{150}{300} \times \frac{75}{300} \right) = \frac{150 \times 75}{300} \\ &= 37.5 \end{aligned}$$

$$\begin{aligned} \text{Female} \cap \text{Red} &= 300 \times \left(\frac{150}{300} \times \frac{75}{300} \right) = \frac{150 \times 75}{300} \\ &= 37.5 \end{aligned}$$

$$\begin{aligned} \text{Male} \cap \text{Blue} &= 300 \times \left(\frac{150}{300} \times \frac{84}{300} \right) = \frac{150 \times 84}{300} \\ &= 42 \end{aligned}$$

$$\begin{aligned} \text{Female} \cap \text{Blue} &= 300 \times \left(\frac{150}{300} \times \frac{84}{300} \right) = \frac{150 \times 84}{300} \\ &= 42 \end{aligned}$$

To find the Chi-Square by hand (only required in the Internal Assessment)... Use the formula

$$\chi^2_{calc} = \sum \frac{(f_o - f_e)^2}{f_e} = \text{sum of } \frac{(\text{given value} - \text{expected value})^2}{\text{expected value}}$$

	Black	White	Red	Blue	Sum
Male	48	12	33	57	150
Female	35	46	42	27	150
Sum	83	58	75	84	300

	Black	White	Red	Blue	Sum
Male	41.5	29	37.5	42	150
Female	41.5	29	37.5	42	150
Sum	83	58	75	84	300

	Black	White	Red	Blue	Sum of all Cells
Male	$\frac{(48 - 41.5)^2}{41.5}$	$\frac{(12 - 29)^2}{29}$	$\frac{(33 - 37.5)^2}{37.5}$	$\frac{(57 - 42)^2}{42}$	33.8
Female	$\frac{(35 - 41.5)^2}{41.5}$	$\frac{(46 - 29)^2}{29}$	$\frac{(42 - 37.5)^2}{37.5}$	$\frac{(27 - 42)^2}{42}$	

$$\chi^2 = 33.8.$$

According to the table from yesterday's notes...

Degrees of freedom (df)	Significance level		
	10%	5%	1%
1	2.71	3.84	6.63
2	4.61	5.99	9.21
3	6.25	7.81	11.34

This number is rather large at any significance level, indicating that gender and t-shirt color are not independent. So we reject the Null Hypothesis.

Note... this is what the calculator does when you enter the contingency table in Matrix A and then do Stats and χ^2 test 😊

Aren't we glad the calculator does all this for us!

Reminder from the last lesson...

To perform the formal χ^2 test of independence, use the following steps:

1. Write the null (H_0) and alternative (H_1) hypotheses.
2. Calculate χ^2_{calc} using your GDC.
3. Determine
 - a. the p -value using your GDC
 - b. the critical value (given on the IB test)
4. Compare
 - a. the p -value against the significance level (given on the IB test)
 - i. If the p -value is less than the significance level, then reject the null hypothesis.
 - ii. If the p -value is more than the significance level, then do not reject the null hypothesis.
 - b. χ^2_{calc} against the critical value
 - i. If χ^2_{calc} is less than the critical value, then do not reject the null hypothesis.
 - ii. If χ^2_{calc} is more than the critical value, then reject the null hypothesis.

Example 1 One hundred people were interviewed outside a chocolate show to find out which flavor of chocolate cream they preferred. The results are given in the table, classified by gender. Perform a χ^2 test, at the 5% significance level, to determine whether preference of chocolate cream flavor and gender are independent.

	Strawberry	Coffee	Orange	Vanilla	Total
Male	23	18	8	8	57
Female	15	6	12	10	43
Total	38	24	20	18	100

- a. State the null hypothesis and the alternative hypothesis

H_0 : Preference of chocolate cream flavor and gender are independent.

H_1 : Preference of chocolate cream flavor and gender are not independent.

- b. Show that the expected frequency for female and strawberry flavor is approximately 16.3.

$$P(\text{female} \cap \text{strawberry}) = P(\text{female}) \times P(\text{strawberry}) = \frac{43}{100} \times \frac{38}{100}$$

So, in this sample of 100 people, we would expect

$$100 \times \left(\frac{43}{100} \times \frac{38}{100} \right) = \frac{43 \times 38}{100} = 16.34$$

- c. Write down the number of degrees of freedom. Remember... don't include the sums!

$$df = (2 - 1)(4 - 1) = 3$$

- d. Write down the χ^2_{calc} value for this data. Use your graphing calc. Go to Stats and then Tests. Use 3 sig figs.

$$\chi^2_{calc} = 6.88$$

Note: You can check your answer to part b after having done the Chi-Square Test. Go to Matrix and Click on [B]. This matrix is now the expected matrix for the given contingency table.

- e. The critical value is 7.815. Using this critical value, comment on your result.

Since χ^2 of 6.88 < 7.815, we fail to reject the null hypothesis. There is sufficient evidence to conclude that the preference of chocolate cream flavor and gender are independent.

- f. Write down the p -value for this data.

$$p\text{-value: } 0.0758$$

- g. Using the 5% significance level (or $\alpha = 0.05$), comment on your result.

Since the p -value of 0.0758 > 0.05, we fail to reject the null hypothesis. There is sufficient evidence to conclude that the preference of chocolate cream flavor and gender are independent.

Example 2 Members of a club are required to register for one of three games: billiards, snooker, or darts. The number of club members of each gender choosing each game in a particular year is shown in the table. Perform a χ^2 test, at the 10% significance level, to determine whether preference of chosen game and gender are independent.

	Billiards	Snooker	Darts	Total
Male	39	16	8	63
Female	21	14	17	52
Total	60	30	25	115

- a. State the null hypothesis and the alternative hypothesis

H_0 : Preference of chosen game and gender are independent.

H_1 : Preference of chosen game and gender are not independent.

- b. Show that the expected frequency for female and billiards is approximately 27.1.

$$P(\text{female} \cap \text{billiards}) = P(\text{female}) \times P(\text{billiards}) = \frac{52}{115} \times \frac{60}{115}$$

So, in this sample of 115 people, we would expect

$$115 \times \left(\frac{52}{115} \times \frac{60}{115} \right) = \frac{52 \times 60}{115} = 27.1$$

If you are comfortable with the steps, you could just show the work in bold.

- c. Write down the number of degrees of freedom.

$$df = (3 - 1)(2 - 1) = 2$$

- d. Write down the χ^2_{calc} value for this data. Use your graphing calc. Use 3 sig figs.

$$\chi^2_{calc} = 7.79$$

- e. The critical value is 4.605. Using this critical value, comment on your result.

Since χ^2 of 7.79 > 4.605, we reject the null hypothesis. There is sufficient evidence to conclude that the preference of chosen game and gender are not independent.

- f. Write down the p -value for this data.

$$p\text{-value: } 0.0203$$

- g. Using the 10% significance level, comment on your result.

Since the p -value of 0.0203 < 0.10, we reject the null hypothesis. There is sufficient evidence to conclude that the preference of chosen game and gender are not independent.