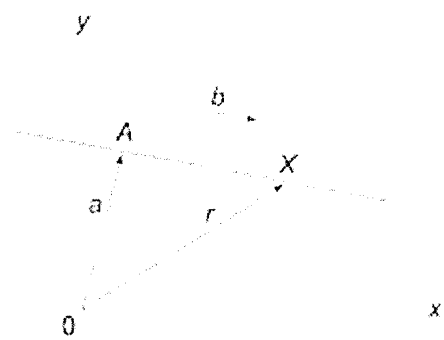
12.4 Vector Equation of a Line

Objectives:

- **Write the vector equation of a line in the form r = a + tb**
- **Use the vector equations of two lines to determine if they are intersecting, parallel, or skew**

Definitions: In the figure at the right, a straight line passes through point *A* such that it is parallel to vector **b**. In addition, **a** represents the position vector for point *A*. *X* represents any point on the line parallel to vector **b**. Since $\overrightarrow{AX}$ is parallel to **b** there is a number *t* such that $\overrightarrow{AX} = t\mathbf{b}$



So, $\mathbf{r} = \overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$

$\mathbf{r} = \mathbf{a} + t\,\mathbf{b}$

where **r** is the <u>vector equation</u> of a line such that **a** is a given <u>position vector</u> of a point on the line and **b** is a <u>directional vector</u> parallel to the line *t* called the parameter.

Example: Find a vector equation of the line that contains (–1, 3, 0) and is parallel to $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

If the line must contain (–1, 3, 0) so that would make $\mathbf{a} = -1\mathbf{i} + 3\mathbf{j} + 0\mathbf{k}$ or $\mathbf{a} = -1\mathbf{i} + 3\mathbf{j}$
If the line must be parallel to $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ that would be the same as **b** in the discussion above.

So $\mathbf{r} = \mathbf{a} + t\,\mathbf{b}$

$\mathbf{r} = -1\mathbf{i} + 3\mathbf{j} + t(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ or in column form $\mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

Example: Find a vector equation of the line passing through *A*(2, 7) and *B* (6, 2)

In this case, we want our line parallel to $\overrightarrow{AB}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ -5 \end{pmatrix} = 4\mathbf{i} - 5\mathbf{j}$ which would be the same as **b** in the discussion above.

So $\mathbf{a} = 2\mathbf{i} + 7\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} - 5\mathbf{j}$

So $\mathbf{r} = \mathbf{a} + t\,\mathbf{b}$

$\mathbf{r} = 2\mathbf{i} + 7\mathbf{j} + t(4\mathbf{i} - 5\mathbf{j})$ or in column form $\mathbf{r} = \begin{pmatrix} 2 \\ 7 \end{pmatrix} + t\begin{pmatrix} 4 \\ -5 \end{pmatrix}$

Two lines $\mathbf{r}_1 = \mathbf{a}_1 + s\,\mathbf{b}_1$ and $\mathbf{r}_2 = \mathbf{a}_2 + t\,\mathbf{b}_2$ with directional vectors $\mathbf{b}_1$ and **$\mathbf{b}_2$** are

- Parallel if $\mathbf{b}_1 = t\,\mathbf{b}_2$
  - If Parallel, you must check to see if they are coincident by seeing if $\mathbf{a}_2$ lies on $\mathbf{r}_1$
- Perpendicular if $\mathbf{b}_1 \bullet \mathbf{b}_2 = 0$

Example: Are the given lines coincident, parallel, or perpendicular?

$$\mathbf{r}_1 = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + s\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \qquad \mathbf{r}_2 = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$$

Parallel?

$$\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = t\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$$

$4 = 1t$ so $t = 4$

$-1 = -2t$ so $t = 0.5$

Since the t's aren't equal, NOT parallel.

Perpendicular?

$$\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = (4)(1) + (-1)(-2) + (2)(-3) = 0$$

Since the product is zero, perpendicular

$$\mathbf{r}_1 = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + s\begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix} \qquad \mathbf{r}_2 = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} -10 \\ 15 \\ -5 \end{pmatrix}$$

Parallel?

$$\begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix} = t\begin{pmatrix} -10 \\ 15 \\ -5 \end{pmatrix}$$

$4 = -10t$ so $t = -0.4$

$-6 = 15t$ so $t = -0.4$

Since the t's are all equal, parallel.

$2 = -5t$ so $t = -0.4$

Now check coincident

$$\begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + t\begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix}$$

$-2 = 2 + 4t$ so $t = -1$

$0 = -1 - 6t$ so $t = -1/6$

Since the t's aren't equal, not coincident.

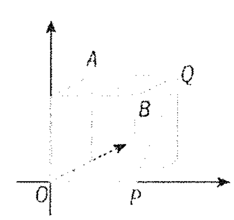
If asked to find whether a given point lies on a given line, set the point equal to the line, as done above when checking coincident.

If asked to find the angle between two lines, use the formula $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \Theta$ where **a** and **b** are the given directional vectors in your two lines.

Homework J

In three dimensions, two lines $\mathbf{r}_1 = \mathbf{a}_1 + s\,\mathbf{b}_1$ and $\mathbf{r}_2 = \mathbf{a}_2 + t\,\mathbf{b}_2$ will either intersect, be parallel, or be skew.

- Intersect: there is a value of t and s such that $\mathbf{r}_1 = \mathbf{r}_2$
- Parallel: $\mathbf{b}_1 = t\,\mathbf{b}_2$
- Skew: if lines don't intersect and they aren't parallel, then they are skew. In the picture at the right, AB and PQ are skew because they will never meet.



Example: Show that the given lines intersect and find the coordinates of their intersection if

$$\mathbf{r}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \mathbf{r}_2 = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -6 \\ -4 \\ 0 \end{pmatrix}$$

If the two vectors are equal if their corresponding components are equal.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \begin{matrix} x = 0 + 1s \\ y = 0 - 1s \\ z = 1 + 1s \end{matrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -6 \\ -4 \\ 0 \end{pmatrix} \qquad \begin{matrix} x = 4 - 6t \\ y = 1 - 4t \\ z = 2 + 0t \end{matrix}$$

$$\begin{aligned} s &= 4 - 6t \\ -s &= 1 - 4t \\ 1 + s &= 2 \end{aligned}$$

So according to the third equation, $s = 1$.

Using substitution on the other two equations gives

$$\begin{aligned} 1 &= 4 - 6t \\ -1 &= 1 - 4t \end{aligned} \qquad \text{so} \quad t = 0.5$$

Note: if you are unable to solve the system, the lines do not intersect.
This could mean they are parallel or skew.

Plug s or t back in to the original vector equation. You should end up with the same result, the intersection point, with either.

$$\mathbf{r}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (1)\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \mathbf{r}_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + (0.5)\begin{pmatrix} -6 \\ -4 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

So the coordinates of the intersection point are (1, –1, 2)

Homework K