12.3 Scalar Product

Objectives:

- **Determine the scalar product of two vectors**
- **Determine the angle between two vectors**
- **Use the scalar product to determine if two vectors are perpendicular, parallel, or coincident**

Definitions: If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ then the <u>scalar product</u> (also known as the dot product) can be found by multiplying the coefficients of i together and the coefficients of j together and adding them up.

Example: If $\mathbf{u} = 8\mathbf{i} - 2\mathbf{j}$ and $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$, find the scalar product.

$$\mathbf{u} \cdot \mathbf{v} = (8)(4) + (-2)(-3) = 32 + 6 = 38$$

Using the Cosine Rule (also known as Law of Cosine) another rule is derived where $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \Theta$ where Θ corresponds to the angle between the two vectors.

If you don't know then angle between two vectors, you could use the above equation or solve for cosine…

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

See the investigation on page 426.

Example: Find the measure of the angle between the 2 vectors $\mathbf{u} = -6\mathbf{i} - 3\mathbf{j}$ and $\mathbf{v} = -8\mathbf{i} + 4\mathbf{j}$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{(-6)(-8) + (-3)(4)}{\sqrt{(-6)^2 + (-3)^2} \cdot \sqrt{(-8)^2 + (4)^2}} = \frac{36}{\sqrt{45} \cdot \sqrt{80}} = \frac{36}{\sqrt{3600}} = \frac{36}{60} = \frac{3}{5}$$

$$\cos\theta = \frac{3}{5}$$

$$\cos^{-1}\left(\frac{3}{5}\right) = 53.13°$$

The formula derived from the cosine rule can also be used to determine whether two vectors are perpendicular (also known as orthogonal), parallel, or coincident (on top of each other)

Perpendicular: Two vectors **a** and **b** that are perpendicular have an angle in between of 90°. Thus they have a scalar product of 0.

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos 90°$$
$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,(0)$$
$$\mathbf{a} \bullet \mathbf{b} = 0$$

Parallel: Two vectors **a** and **b** that are parallel have an angle in between of 0° or 180°. Thus they have a scalar product of $\pm\,|\mathbf{a}|\,|\mathbf{b}|$

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos 0°$$
$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,(1)$$
$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|$$

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos 180°$$
$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,(-1)$$
$$\mathbf{a} \bullet \mathbf{b} = -\,|\mathbf{a}|\,|\mathbf{b}|$$

***Alternately, vectors that are parallel are scalar multiples of the other. For example, $3\mathbf{i} - 2\mathbf{j}$ and $6\mathbf{i} - 4\mathbf{j}$

Coincident: Vectors that coincide are essentially the same vector, thus have an angle in between of 0°

$$\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|\,|\mathbf{a}| \cos 0°$$
$$\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|\,|\mathbf{a}|\,(1)$$
$$\mathbf{a} \bullet \mathbf{a} = \mathbf{a}^2$$

Example: Determine whether **u** and **v** are orthogonal, parallel, or neither if $\mathbf{u} = 0.25\,(3\mathbf{i} - \mathbf{j})$ and $\mathbf{v} = 5\mathbf{i} + 6\mathbf{j}$

First, $\mathbf{u} = 0.75\mathbf{i} - 0.25\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} + 6\mathbf{j}$

$$\mathbf{u} \bullet \mathbf{v} = (0.75)(5) + (-0.25)(6) = 3.75 + -1.5 = 2.25$$

Since the scalar product is neither 0 nor 1, the two vectors are neither orthogonal nor parallel.

Example: Find the value of k such that **u** and **v** are orthogonal if $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} - k\mathbf{j}$

Since the vectors are orthogonal, the angle in between is 90° so as discussed earlier, they should have a scalar product of 0.

$$\mathbf{u} \bullet \mathbf{v} = (3)(2) + (2)(-k) = 6 + -2k$$

$$6 + -2k = 0$$

$$k = 3$$

Homework I