12.1 Vectors: basic concepts

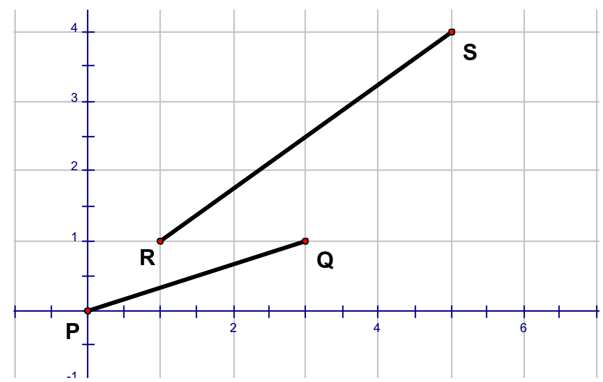
Background: Vectors are part of a branch of Physics called Mechanics. They are used to represent quantities such as displacement, force, weight, velocity, and momentum. In Mathematics the focus is primarily on displacement (the difference between the initial and final position) and velocity (the rate at which something changes its position).

Definitions: A scalar has size (magnitude) but not direction. We think of real numbers as scalars, even if they are negative. For example, a speed such as 55 mph is a scalar quantity.

A vector has both size (magnitude) and direction. For example, a velocity of 55 mph due North is a scalar vector. The length of a vector indicates its magnitude and the arrow indicates its direction.

Notation: A vector **v** is written as v̲ if you can't write in boldface or with two letters indicating the direction of the movement such as $\overrightarrow{RS}$ (from R to S).

Vectors have a horizontal and a vertical component. For example, to describe the movement from R to S we would say 'move 4 units in the positive x direction (Horizontal Component) and 3 units in the positive y direction (Vertical Component). The direction and length of the movement is important.



A vector can be written in two forms, Column Vector Form (similar to component form) or Unit Vector Form

Column Vector Form

$\begin{pmatrix} x \\ y \end{pmatrix}$ where x is the horizontal component and y is the vertical component

$$\mathbf{v} = \underline{v} = \overrightarrow{RS} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

Unit Vector Form

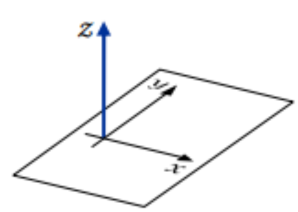
$x\mathbf{i} + y\mathbf{j}$ where x is the horizontal component and y is the vertical component and **i** and **j** are base vectors of length one in the direction of x and y respectively.

$$\mathbf{v} = \underline{v} = \overrightarrow{RS} = 4\mathbf{i} + 3\mathbf{j}$$

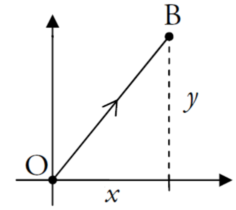
FYI… Component Form would be <4, 3>

In three dimensions a third component **k** represents a vector of length one in the z-direction.

For example, $\mathbf{a} = \underline{a} = \overrightarrow{AB} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} = -2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$



*The <u>magnitude</u> (size, modulus, or norm) of a vector is the length of the vector. It can be found by using the Pythagorean Theorem.

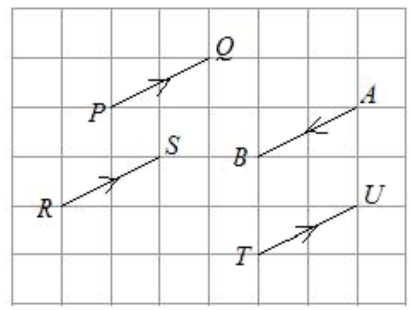


$$|\mathbf{v}| = |\underline{v}| = \left|\overrightarrow{OB}\right| = \sqrt{x^2 + y^2}$$

In three-dimensions $|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$

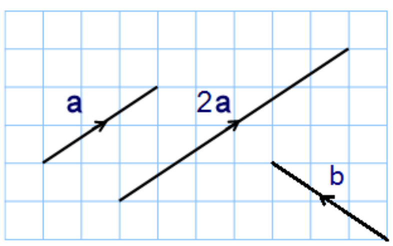
Homework A

*Two vectors are <u>equal</u> if they have the same direction and magnitude. They have the same column and unit vector forms. All the vectors at the right are equal except $\overrightarrow{AB}$



$\overrightarrow{AB}$ is a <u>negative vector</u> to all the rest.

Since $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{AB} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$, $\overrightarrow{-AB} = \overrightarrow{PQ}$



*Vectors are <u>parallel</u> if one is a scalar multiple of the other. All the vectors above are parallel. In the figure at the left,

$a = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $2a = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$. They are parallel using a scalar of 2

a and b have the same magnitude but different direction.

$a = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $b = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ so they are not parallel.

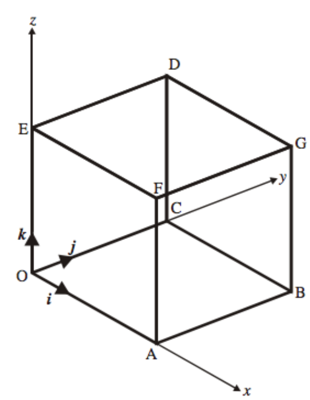
Example: For what values of x and y are the vectors $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + x\mathbf{k}$ and $\mathbf{b} = y\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ parallel?

So since **a** and **b** are parallel, $\mathbf{a} = k\mathbf{b}$ where k is a scalar.

So $2\mathbf{i} - \mathbf{j} + x\mathbf{k} = k(y\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$
So $2 = ky$; $-1 = 2k$; $x = -3k$

I can solve the middle equation for k... $k = -0.5$ so $2 = (-0.5)y$ and $x = -3(-0.5)$
So $y = -4$ and $x = 1.5$

Example: The diagram at the right shows a cube, OABCDEFG, where the length of each edge is 5 cm. Express the following vectors in terms of **i**, **j**, and **k**.

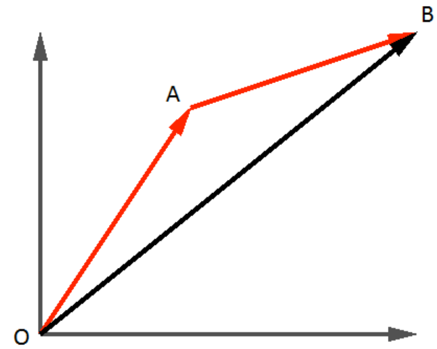


a) $\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BG} = 5\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$

b) $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = -5\mathbf{i} + 5\mathbf{k}$

c) $\overrightarrow{EB} = \overrightarrow{EF} + \overrightarrow{FG} + \overrightarrow{GB} = 5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}$

Homework B



*Position vectors give the position of a point relative to the origin, O. In the figure at the right if the point A is (3, 4) and B is (7, 5) then they have position vectors

$\overrightarrow{OA} = \begin{pmatrix}3\\4\end{pmatrix} = 3\mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{OB} = \begin{pmatrix}7\\5\end{pmatrix} = 7\mathbf{i} + 5\mathbf{j}$

*You could go from A to B by going directly from A to B or first going from A to O and then from O to B.

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$

$\overrightarrow{AB}$ is called the resultant vector of $\overrightarrow{AO}$ and $\overrightarrow{OB}$

Since $\overrightarrow{AO} = \overrightarrow{-OA}$, $\overrightarrow{AB} = \overrightarrow{-OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$

Thus $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

Example:

Using the information above, find $\overrightarrow{AB}$

Draw a picture like the one above if you need to. If the vectors you are given are not position vectors (ones coming from the origin), you can still do it the same way...

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{-OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA} =$$

$$\begin{pmatrix}7\\5\end{pmatrix} - \begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}4\\1\end{pmatrix} = 4\mathbf{i} + \mathbf{j}$$

Example: Given that $\overrightarrow{XY} = \begin{pmatrix}1\\-3\end{pmatrix}$ and $\overrightarrow{YZ} = \begin{pmatrix}0\\-5\end{pmatrix}$ find $\overrightarrow{XZ}$

Since I am going from X to Y to Z, I can simply add the vector components...

so $\overrightarrow{XZ} = \begin{pmatrix}1\\-3\end{pmatrix} + \begin{pmatrix}0\\-5\end{pmatrix} = \begin{pmatrix}1\\-8\end{pmatrix}$

Now find $\overrightarrow{ZX}$

Since it is just going in the opposite direction of $\overrightarrow{XZ}$, just find the opposite vector...

$$\overrightarrow{XZ} = -\overrightarrow{XZ} = -1\begin{pmatrix}1\\-8\end{pmatrix} = \begin{pmatrix}-1\\8\end{pmatrix}$$

Homework C

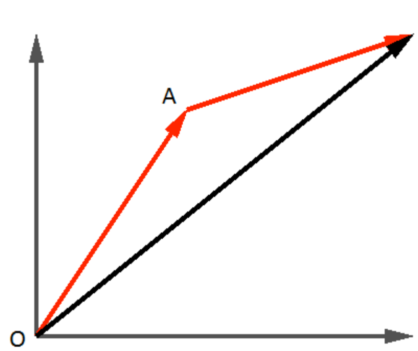
*Vectors can be used to show 3 points A, B, and C are collinear (all on the same line).

For ex, find $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and show that they are parallel (scalar multiples) and share a common point.

Example: If $\overrightarrow{AB} = 3\mathbf{i} - 4\mathbf{j}$ and $\overrightarrow{AC} = -6\mathbf{i} + 8\mathbf{j}$ then A, B, and C are collinear since $\overrightarrow{AC} = -2\overrightarrow{AB}$

Homework D

*Vectors can be used to find the distance between two points in two or three dimensions.



To find the distance from A to B, find

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$$

$$\overrightarrow{AB} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example: Find the vector from A (2, –3, 5) to B (4, 7, –6)

$\overrightarrow{OB} = \mathbf{b} = 4\mathrm{i} + 7\mathbf{j} - 6\mathbf{k}$ and $\overrightarrow{OA} = \mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (4\mathrm{i} + 7\mathbf{j} - 6\mathbf{k}) - (2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) = 2\mathbf{i} + 10\mathbf{j} - 11\mathbf{k}$

$\left|\overrightarrow{AB}\right| = \sqrt{(2)^2 + (10)^2 + (-11)^2} = \sqrt{4 + 100 + 121} = \sqrt{225} = 15$

Homework E

*A unit vector is a vector of length 1 in a given direction. To find one, take the vector **a** and divide by the length |**a**|: $\frac{\mathbf{a}}{|\mathbf{a}|}$

*To find a vector in the same direction of length k, multiply the unit vector times k: $k\frac{\mathbf{a}}{|\mathbf{a}|}$

Example: Find a unit vector in the same direction as the vector $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ (or $\mathbf{a} = -3\mathbf{i} + 2\mathbf{j}$).

Then find a vector of length 5 in the same direction.

$|\mathbf{a}| = \sqrt{(-3)^2 + (2)^2} = \sqrt{13}$ so $\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{\sqrt{13}}(-3\mathbf{i} + 2\mathbf{j}) = \frac{\sqrt{13}}{13}(-3\mathbf{i} + 2\mathbf{j}) = \frac{-3\sqrt{13}}{13}\mathbf{i} + \frac{2\sqrt{13}}{13}\mathbf{j}$

Since the unit vector has a length of unit one, I can take its vector times 5.

$5\left(\frac{-3\sqrt{13}}{13}\mathbf{i} + \frac{2\sqrt{13}}{13}\mathbf{j}\right) = \frac{-15\sqrt{13}}{13}\mathbf{i} + \frac{10\sqrt{13}}{13}\mathbf{j}$

Homework F