1 Prior Learning Topics

Overview

The purpose of this chapter is to review topics that you are expected to be familiar with. These topics include: numbers, fractions, algebra, geometry, trigonometry, financial mathematics and statistics. Questions from the IB examinations will assume knowledge of these topics. Therefore, you should refer to these topics, concepts and solutions as needed, as you work through the text.

Each section will provide the basic concepts and building blocks necessary for mastery of the forthcoming chapters.

1.1 Numbers

Classifying numbers

The idea of number is older than recorded history. Mankind's ability to count probably began with the necessity for recording quantities, using tally bones. The oldest example may be the piece of baboon leg bone showing 29 notches or tally marks. This was found in Swaziland and is believed to date back to 35 000 BCE

There have been many number systems used throughout history. A few of these are: Egyptian, Babylonian, Greek, Roman, Chinese-Japanese, Mayan, and Hebrew systems. Today we use the Hindu-Arabic number system. This system is named after the Hindus, who invented it, and the Arabs who introduced it to western civilization.

The numbers in the Hindu-Arabic system can be classified according to the properties that each has. Listed below are the classifications (sets) you will need to know.

Natural numbers $= \mathbb{N} = \{0, 1, 2, 3, \ldots\}$
Integer numbers $= \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
Positive integer numbers $= \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$
Rational numbers $= \mathbb{Q} =$ numbers that can be expressed as a ratio of two integers $\left(\frac{p}{q} : q \neq 0\right)$.

$$Q = \left\{\begin{array}{l} \ldots, -\frac{1}{3}, -\frac{1}{2}, -\frac{1}{1}, 0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \\ \ldots, -\frac{2}{3}, -\frac{2}{2}, -\frac{2}{1}, \quad \frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \ldots \\ \ldots, -\frac{3}{3}, -\frac{3}{2}, -\frac{3}{1}, \quad \frac{3}{1}, \frac{3}{2}, \frac{3}{3}, \ldots \end{array}\right\}$$

Positive rational numbers $= \mathbb{Q}^+ = \{x | x \in \mathbb{Q}, x > 0\}$
Irrational numbers $= \mathbb{Q}' = \{$real numbers that are not rational$\}$
Real numbers $= \mathbb{Q} \cup \mathbb{Q}' = \{$all numbers on the number line$\}$
Positive real numbers $= \mathbb{R}^+ = \{x | x \in \mathbb{R}, x > 0\}$

$\{x | x \in \mathbb{Q}, x > 0\}$
This notation is read: 'the set of all x, such that x is an element of the set of rational numbers and x is greater than zero.'

A Venn diagram is often helpful in visualizing the relationships between sets of numbers.

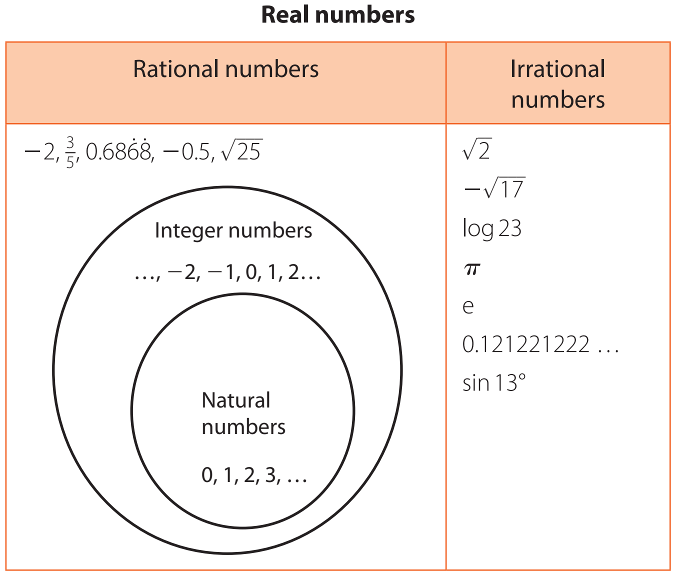


Figure 1.1 Venn diagram for the set of real numbers.

There are exactly the same number of numbers in the set of natural numbers as there are in the set of integer numbers or even the set of rational numbers! However, there are more real numbers than there are natural numbers! There is more than one level of infinity!

What are some of the ways that mankind has dealt with the concept of *infinite*? Exactly what does it mean to be infinitely large?

Example 1.1

Classify each of the following numbers as $\mathbb{N}, \mathbb{Z}, \mathbb{Z}^+, \mathbb{Q}, \mathbb{Q}^+, \mathbb{Q}', \mathbb{R}, \mathbb{R}^+$.

a) 3 b) 0.4 c) $\sqrt{2}$ d) $-\frac{37}{5}$

Solution

a) Since $3 = \frac{3}{1}$, 3 is a member of $\mathbb{N}, \mathbb{Z}, \mathbb{Z}^+, \mathbb{Q}, \mathbb{Q}^+, \mathbb{R}$, and $\mathbb{R}^+$.

b) Since $0.4 = \frac{4}{10}$, 0.4 is a member of $\mathbb{Q}, \mathbb{Q}^+, \mathbb{R}$, and $\mathbb{R}^+$.

c) Since $\sqrt{2}$ cannot be expressed as the ratio of two integers, $\sqrt{2}$ is a member of $\mathbb{Q}', \mathbb{R}$, and $\mathbb{R}^+$.

d) $-\frac{37}{5}$ is located in the 37th column to the left of 0 and down to the 5th row of the chart on page 1. Therefore, $-\frac{37}{5}$ is a member of $\mathbb{Q}$ and $\mathbb{R}$.

To show that a number is irrational requires the study of number theory.

For the proof that $\sqrt{2}$ is irrational, visit www.pearsonhotlinks.co.uk, enter the ISBN for this book and click on weblink 1.1.

Order of operations

In order to avoid confusion when performing a series of arithmetic operations, we follow a standard order:

Step 1: Eliminate all **P**arentheses.

Step 2: Simplify all **E**xponents.

Step 3: Perform **M**ultiplication and **D**ivision as you come to them, reading from left to right.

Step 4: Perform **A**ddition and **S**ubtraction as you come to them, reading from left to right.

An easy way to remember these steps is by using the mnemonic:
Please **E**xcuse **M**y **D**ear **A**unt **S**ally (**PEMDAS**)

Example 1.2

Simplify each of the following.

a) $3 \cdot 5 - 7 + 8 \div 2$

b) $2(9 - 5)^2 + -1 \times 10 \div 2$

c) $\sqrt{4^2 + 3^2}$

Solution

a) $3 \cdot 5 - 7 + 8 \div 2 = 15 - 7 + 4 = 8 + 4 = 12$

b) $2(9 - 5)^2 + -1 \times 10 \div 2 = 2(4)^2 + -10 \div 2 = 2 \cdot 16 + -5$
$= 32 + -5 = 27$

c) $\sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$

The answer to part c) is not ± 5 since $\sqrt{25}$ is asking for the principal square root, which, by definition, is always positive.

Number theory

Number theory is the study of integers. This review section will cover the basics: prime numbers, factors and multiples.

Prime numbers

A prime number is defined as a natural number greater than 1 whose only positive divisors are 1 and itself.

For example, 5 is a prime number since the only natural numbers (other than 0) that divide into it (without a remainder) are 1 and 5.

As a counter-example, 6 is not a prime number, since both 2 and 3 divide into 6.

All other natural numbers (0 and 1 excluded) that are not prime are called **composite**.

By definition, the first prime number is 2.

Below is a partial list of prime numbers.

Prime numbers $= \{2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\}$

Some books on number theory include proofs that show there are infinitely many prime numbers, that there are infinitely many levels of infinite, that the square root of any prime number is not a rational number and that π is irrational.

That there are infinitely many primes is not a foregone conclusion since, as you think of larger and larger numbers, there are also more and more numbers that have a chance of dividing into that number, thus making that large number not prime!

Example 1.3

Determine if 137 is a prime number.

Solution

Start dividing 137 by natural numbers to see if any divide into 137 without a remainder. $\frac{137}{2} = 68.5$

$\frac{137}{3} = 45.\overline{6}$ and so on.

Do this procedure using all of the natural numbers that are smaller than or equal to the $\sqrt{137}$ plus one (i.e. $11.7 + 1 = 12$).

$\frac{137}{12} = 11.41\overline{6}$

Since none of those natural numbers divide 137 evenly, 137 must be prime.

You only have to test the numbers up to the approximate value of $\sqrt{137}$ since 12×12 is greater than 137.

Factors

Factors are numbers that are multiplied together.

In this book, factors are considered to be natural numbers other than 0.

For example, 2 and 5 are called factors of 10 since $2 \cdot 5 = 10$.

A commonly worded question is to ask for a natural number to be written as a product of primes, often called the **prime factorization** of that number.

Example 1.4

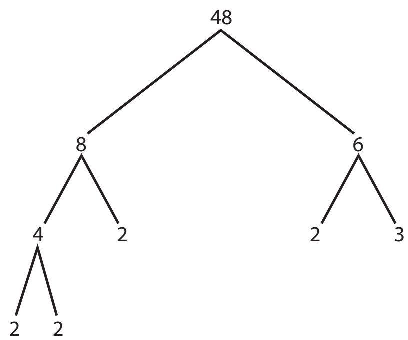
Write 48 as a product of primes.

Solution

$$\begin{aligned} 48 &= 2 \cdot 24 \\ &= 2 \cdot 2 \cdot 12 \\ &= 2 \cdot 2 \cdot 2 \cdot 6 \\ &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \\ &= 2^4 \cdot 3 \end{aligned}$$

Another popular technique used is called the **factor tree**.

Figure 1.2 Writing the prime factorization of 48 using a factor tree.



Example 1.5

List all of the natural number factors of 45.

Solution

The factors of 45 are 1, 3, 5, 9, 15, 45 since those are the only natural numbers to divide into 45 (without a remainder).

The greatest common factor (GCF) of two or more natural numbers is the greatest number that will divide into all of the given numbers.

Example 1.6

Find the greatest common factor of 24 and 36.

A notation that can be used to express the idea of the greatest common factor is 'GCF(24, 36)'.

Solution

The factors of 24 are 1, 2, 3, 4, 6, 8, **12**, 24.

The factors of 36 are 1, 2, 3, 4, 6, 9, **12**, 18, 36.

As you can see, there are five common factors for 24 and 36, but there is only one that is the largest, the greatest. Therefore, the GCF(24, 36) = 12.

Multiples

A **multiple** of a natural number is the product of that number and another natural number.

There are infinitely many multiples of natural numbers (except 0).

For example, 30 is a multiple of 3 since $3 \cdot 10 = 30$.

Example 1.7

List the first five multiples of each of the following.

a) 3
b) 17
c) 100

Solution

a) $3 \cdot 1 = 3, 3 \cdot 2 = 6, 3 \cdot 3 = 9, 3 \cdot 4 = 12, 3 \cdot 5 = 15$
b) Another method involves adding 17 to each preceding number:
$17, 17 + 17, 34 + 17, 51 + 17, 68 + 17 \Rightarrow 17, 34, 51, 68, 85$
c) The first five multiples of 100 are 100, 200, 300, 400, 500.

Using the TI Calculator, repeated addition can be completed as shown.

```
17
                17
Ans+17
                34
Ans+17
                51
```

The least common multiple (LCM) of two or more numbers is the smallest number that the given numbers will divide into.

Example 1.8

Find the least common multiple of 12 and 18.

Solution

The multiples of 12 are: 12, 24, **36**, 48, 60, 72, 84, 96, 108, …

The multiples of 18 are: 18, **36**, 54, 72, 90, 108, …
As you can see, there are infinitely many common multiples, but there is only one that is the smallest, the least. Therefore, the LCM(12, 18) = 36.

Using the TI Calculator, the LCM can be found as shown.

```
lcm(12.18)
                36
```

Ratios, percentages and proportions

A **ratio** is a quotient (a fraction) of two numbers.

A **percentage** is a part of 100.

A **proportion** is an equation involving two ratios.

An example of a ratio is $\frac{250}{30}$. You can think of this ratio as 250 for every 30. We can apply ratios in our everyday lives. This ratio could represent 250 kilometres for every 30 litres or it could represent 250 apples for every 30 horses. Some ratios can be simplified in order to make large numbers less cumbersome. $\frac{250}{30}$ can be simplified by dividing both the numerator (the top number) and the denominator (the bottom number) by the GCF(30, 250). Since 10 is the GCF, then $\frac{250}{30} = \frac{250 \div 10}{30 \div 10} = \frac{25}{3}$.

The International System of Units is abbreviated to SI. This is from the French phrase Le Système International d'unités.

Can you see the number 100 disguised as the '%' symbol?

An example of a percentage would be 20 per cent or 20%.

20% can be written as the ratio $\frac{20}{100}$.

An example of a proportion would be $\frac{20}{100} = \frac{1}{5}$. Sometimes a proportion is written in this style: $20:100 = 1:5$ where 20 and 5 are called the **extremes** and 100 and 1 are called the **means**. You should be able to see that $20 \times 5 = 100 \times 1$. Putting the concept of proportions into words we say, 'the product of the extremes is the product of the means'. This idea can be written as a general rule.

The proof, or verification, of this rule involves multiplying both sides of the proportion by the product of the numbers in the denominators.

$$\text{If } \frac{a}{b} = \frac{c}{d} \text{ then } a \cdot d = c \cdot b\text{, where } b, d \neq 0.$$

Example 1.9

Consider the fact that 5280 feet = 1.609 kilometres.

a) Write a ratio for feet to kilometres.

b) Write a proportion using 10 560 feet and 3.218 kilometres.

c) Write a proportion describing 30% of 5280 feet.

Solution

a) $\frac{5280}{1.609}$ b) $\frac{5280}{1.609} = \frac{10\,560}{3.218}$ c) $\frac{30}{100} = \frac{x}{5280}$

Exercise 1.1

1. List the set of integers between -5 and 3 inclusive.

2. Using the chart on page 1, describe where the rational number $\frac{16}{27}$ can be located.

3. Using the chart on page 1, write the fourth row for the set of rational numbers.

4. What is the first integer number to the right of 0 on a number line?

5. What is the first rational number to the right of 0 on a number line?

6. What is the first real number to the right of 0 on the number line?

7. List the set of natural numbers that are multiples of 2.

8. List the set of natural numbers that are one more than the multiples of 2.

9. List the first twenty prime numbers.

10. Classify each of the following as $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Q}$, $\mathbb{Q}^+$, $\mathbb{Q}'$, $\mathbb{R}$, $\mathbb{R}^+$, prime.

a) 2 b) -5 c) $\frac{2}{3}$ d) $\sqrt{5}$ e) 10^{50}

f) π g) $\frac{22}{7}$ h) 0 i) 1 j) -1.23×10^{-3}

11. List the positive integer factors of:

a) 18 b) 45 c) 72 d) 100

12. List the first five positive multiples for:

a) 3 b) 13 c) 19

13. Write each of the following as a product of primes:

a) 108 b) 75 c) 5733

14. Determine if each of the following is prime:

a) 253 b) 257 c) 391 d) 421

15. Find the GCF and LCM of each of the following:

a) 36, 48 b) 18, 24, 100 c) 9, 12, 33

16. Simplify each of the following as an exact answer or as indicated:

a) $(-2)^2 \cdot 3 + 5 \div 4$

b) $-2^2 \times 4^{-1} + 6 \times 2 - 1$

c) $4000\left(1 + \frac{0.056}{12}\right)^{12 \cdot 10}$, correct to 2 decimal places

d) $\frac{1}{\sin 23° - \sin 37°}$, correct to 2 decimal places

e) $7 + 6 \div 2 - 3 \times 4 + 4 \div 2 + 6$

f) $2 + 3^{-1} - \frac{\log 13}{\log 2}$, correct to 2 decimal places

g) $\sqrt{15^2 - 5^2}$

h) $4 \div 2 \times 3$

17. Consider the fact that 720 feet = 120 fathoms.

a) Write a ratio for feet to fathoms.

b) Write a proportion using 2160 feet and 360 fathoms.

c) Write a proportion describing 90% of 360 fathoms.

18. Consider the fact that 12 months = 365.25 days.

a) Write a ratio for months to days.

b) Express that ratio as the simplified quotient of two integers.

c) Write a proportion using 48 months and correct number of days.

d) Express that ratio as the simplified quotient of two integers.

e) Write a proportion describing 25% of 48 months.

1.2 Fractions

In general, a fraction is the quotient of any two quantities. Another name for a fraction is a ratio. For example, $\frac{\text{miles}}{\text{hour}}$ is the fraction (a ratio) expressing a distance in miles and time in hours. In arithmetic, a simple fraction is usually considered as the quotient of two integer numbers such as $\frac{2}{3}$. There are many types of fractions: complex, similar, rational, proper, improper, terminating, non-terminating, continued, partial and repeating, to name a few. Fractions can be added, subtracted, multiplied, divided, simplified, changed to decimals, and so on. This section will review the important properties of simple fractions.

• **Examiner's hint:** When taking the IB exams you may leave the fraction in unsimplified form, e.g. $\frac{6}{8}$, and still receive full credit. However, you should always try to simplify to lowest terms.

Simplifying fractions

The 'top' number of a fraction is the numerator. The 'bottom' number is called the denominator.

A **proper fraction** has a numerator that is less than its denominator.

An **improper fraction** has a numerator that is larger than its denominator.

A simple fraction is said to be simplified if the GCF (numerator, denominator) = 1. For example, $\frac{4}{7}$ is said to be simplified since GCF (4,7) = 1.

When the greatest common factor of two numbers, such as 4 and 7, is equal to one, then those numbers are said to be relatively prime.

Example 1.10

Simplify $\frac{12}{18}$

Solution 1

Since the GCF(12, 18) = 6, then $\frac{12}{18} = \frac{2 \cdot 6}{3 \cdot 6} = \frac{2}{3}$.

Solution 2

$$\frac{12}{18} = \frac{12 \div 6}{18 \div 6} = \frac{2}{3}$$

The keystrokes ALPHA, Y = , 1, 12, >, 18, ENTER will simplify the unsimplified fraction.

Solution 3

Press the following keys on your calculator:

ALPHA, Y = , 1, 12, >, 18, ENTER

The simplified fraction is shown.

Multiplying fractions

When multiplying one fraction by another, multiply the numerators and multiply the denominators.

Example 1.11

Find the product of $\frac{5}{9} \cdot \frac{4}{7}$

Solution 1

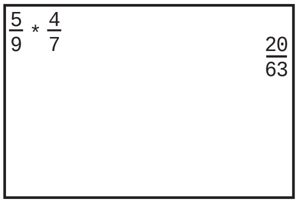
$$\frac{5}{9} \cdot \frac{4}{7} = \frac{5 \cdot 4}{9 \cdot 7} = \frac{20}{63}$$

Solution 2

Press the following keys on your calculator:

ALPHA, Y = , 1, 5, >, 9, >, ×, ALPHA, Y = , 1, 4, > 7, ENTER

The answer is as shown.



When multiplying a fraction by an integer, rewrite the integer as a fraction whose numerator is the integer and whose denominator is 1 and then follow the above rules.

For example, $5 = \frac{5}{1}$

Example 1.12

Find the product of $\frac{5}{8} \cdot 3$, leaving your answer as an improper fraction.

Solution 1

$$\frac{5}{8} \cdot 3 = \frac{5}{8} \cdot \frac{3}{1} = \frac{5 \cdot 3}{8 \cdot 1} = \frac{15}{8}$$

Solution 2

Press the following keys on your calculator:

ALPHA, Y = , 1, 5, >, 8, >, ×, 3, ENTER

The answer is as shown.

$\frac{5}{8} * 3$

$\frac{15}{8}$

Dividing fractions

When dividing two fractions, multiply the first by the reciprocal of the second.

Example 1.13

Find the quotient of $\frac{3}{7} \div \frac{11}{5}$

Solution 1

$$\frac{3}{7} \div \frac{11}{5} = \frac{3}{7} \cdot \frac{5}{11} = \frac{3 \cdot 5}{7 \cdot 11} = \frac{15}{77}$$

Solution 2

Press the following keys on your calculator:

ALPHA, Y = , 1, 3, >, 7, >, ÷, ALPHA, Y = , 1, 11, >, 5, ENTER

The answer is as shown.

$\frac{3}{7} / \frac{11}{5}$

$\frac{15}{77}$

A **complex fraction** is a fraction in which both the numerator and denominator are simple fractions.

For example, $\dfrac{\frac{4}{5}}{\frac{7}{9}}$

When simplifying a complex fraction, treat it as the division of two fractions: the numerator times the reciprocal of the denominator.

Example 1.14

Simplify the complex fraction $\dfrac{4/5}{7/9}$

Solution

$$\frac{4/5}{7/9} = \frac{4}{5} \cdot \frac{9}{7} = \frac{4 \cdot 9}{5 \cdot 7} = \frac{36}{35}$$

It is important to know the process for adding and subtracting fractions for algebra, as in Exercise 1.2.

Adding and subtracting fractions

Fractions that are similar have common or like denominators.

Finding the sum or difference of two fractions requires that the fractions be similar.

When adding fractions that are similar, add the numerators and keep the same denominator.

When subtracting two similar fractions, subtract the numerators and keep the same denominator.

Example 1.15

Find the sum of $\frac{2}{3} + \frac{4}{5}$, leaving the answer as an improper fraction.

Multiplying $\frac{2}{3}$ by $\frac{5}{5}$ did not change its value. It just 'unsimplified' it.

Solution 1

$$\frac{2}{3} + \frac{4}{5} = \frac{2}{3} \cdot \frac{5}{5} + \frac{4}{5} \cdot \frac{3}{3} = \frac{10}{15} + \frac{12}{15} = \frac{10 + 12}{15} = \frac{22}{5}$$

Solution 2

Press the following keys on your calculator.

ALPHA, Y = , 1, 2, 3, >, +, ALPHA, Y = , 1, 4, >, 5, ENTER

The answer is as shown.

$$\frac{2}{3} + \frac{4}{5} \qquad \frac{22}{15}$$

Example 1.16

Find $\frac{1}{3} - \frac{7}{10}$.

Solution 1

$$\frac{1}{3} - \frac{7}{10} = \underbrace{\frac{1}{3} \cdot \frac{10}{10} - \frac{7}{10} \cdot \frac{3}{3} = \frac{1 \cdot 10}{3 \cdot 10} - \frac{7 \cdot 3}{10 \cdot 3} = \frac{10}{30} - \frac{21}{30}}_{\text{These steps will be skipped in Solution 2, the 'cross-multiply' method.}} = \frac{10 - 21}{30} = \frac{-11}{30}$$

Solution 2

$$\frac{1}{3}-\frac{7}{10}=\underbrace{\frac{1\cdot 10-3\cdot 7}{3\cdot 10}}_{\text{Even this step can be skipped!}}=\frac{10-21}{30}=\frac{-11}{30}$$

Exercise 1.2

1. Simplify each of the following:
   a) $\frac{24}{36}$ b) $\frac{75}{125}$ c) $\frac{512}{128}$ d) $\frac{255}{153}$
2. Find each product or quotient. Leave your answer as a proper or improper fraction in simplified form.
   a) $\frac{3}{4}\cdot\frac{6}{7}$ b) $\frac{3}{4}\div\frac{6}{7}$ c) $\frac{64}{200}\cdot -30$ d) $16\cdot\frac{5}{8}$ e) $\frac{-12/28}{3/7}$
   f) $\frac{72}{96}\div\frac{21}{12}$ g) $\frac{a}{b}\cdot\frac{c}{d}$ h) $\frac{e}{f}\div\frac{g}{h}$ i) $2\div\frac{1}{2}$ j) $\frac{5}{8}\div 4$
3. Find each sum or difference. Express your answer as a proper or improper fraction in simplified form.
   a) $\frac{1}{2}+\frac{3}{5}$ b) $\frac{7}{11}-\frac{3}{4}$ c) $6+\frac{4}{5}$ d) $-\frac{3}{10}-3$ e) $2\frac{3}{5}+4\frac{7}{10}$
   f) $12\frac{1}{3}-15\frac{7}{9}$ g) $\frac{a}{b}+\frac{c}{d}$ h) $\frac{x}{y}-\frac{y}{x}$ i) $z-\frac{1}{y}$ j) $1-\frac{1}{x}$

1.3 Algebra

Expanding and factorizing

An axiom is a fundamental statement we assume is true without proof. We **must** accept some statements as true or other statements, called theorems, would not be possible to prove. One such statement is called the **distributive axiom**. It is one of the eleven fundamental statements called the **field axioms**.

The distributive axiom states:

$$a(b+c)=ab+ac, \text{ for all } a, b, c \in \mathbb{R}.$$

You should recall that when reading a mathematics equation you must read from left to right as well as from right to left.

For example, reading and applying the axiom from left to right we have: $3(4+5)=3\cdot 4+3\cdot 5$. Reading and applying the axiom in this manner is often called expanding.

Conversely, reading and applying the axiom from right to left we have: $7\cdot 9+7\cdot 13=7(9+13)$. Reading and applying the axiom in this manner is often called factorizing.

Since a variable, such as x, is merely a symbol used to represent a real number, the distributive axiom continues to hold true when they are used. For example, $5(x+7)=5x+5\cdot 7=5x+35$.

W For a historical perspective of algebra, visit www.pearsonhotlinks.co.uk, enter the ISBN for this book and click on weblink 1.2.

W For an entire list of the field axioms, visit www.pearsonhotlinks.co.uk, ISBN for this book and click on weblink 1.3.

Expanding

Terms are expressions that are being added or subtracted.

Factors are expressions that are being multiplied.

You can think of **expanding** as 'going' from one term to many terms.

An expression such as $(x + 2)(x + 3)$ is considered as one term with two factors.

Example 1.17

Expand each of the following using the distributive axiom. Do not simplify your answer.

a) $6(4 + 8)$ b) $-4(x + 3)$ c) $\sqrt{2}(y - 7)$

Solution

a) $6(4 + 8) = 6 \cdot 4 + 6 \cdot 8$

b) $-4(x + 3) = -4x + -4 \cdot 3$

c) $\sqrt{2}(y - 7) = \sqrt{2}(y + -7) = \sqrt{2}y + \sqrt{2} \cdot (-7)$

Polynomials are expressions having more than one term. For example, $x^2 + 5x + 6$ is a polynomial with three terms. When expanding the product of two binomials (a polynomial with two terms), the distributive axiom may still be used.

Example 1.18

Expand and simplify the product $(x + 2)(x + 3)$.

Solution

Think of the $(x + 2)$ as the 'a' in the axiom and x and 3 as 'b' and 'c' respectively. Therefore,

$$\begin{aligned}(x + 2)(x + 3) &= (x + 2) \cdot x + (x + 2) \cdot 3 \\ &= x \cdot (x + 2) + 3 \cdot (x + 2) \\ &= x \cdot x + x \cdot 2 + 3 \cdot x + 3 \cdot 2 \\ &= x^2 + 2x + 3x + 6 \\ &= x^2 + 3x + 2x + 6 \\ &= x^2 + 5x + 6\end{aligned}$$

See the commutative axiom for multiplication in the field axiom list.

Commuting $2x$ and $3x$ is important for understanding the shortcut for expanding the product of two binomials. However, it is not necessary to include this step.

The above solution suggests a shorter method for finding the product. The mnemonic is **FOIL**. The letters stand for: **F**irst, **O**utside, **I**nside, **L**ast.

In other words, multiply the **F**irst terms x and x, and then the **O**utside terms x and 3, and then the **I**nside terms 2 and x, and finally the **L**ast terms 2 and 3.

Example 1.19

Expand and simplify $(x + 4)(x + 7)$ using the FOIL method.

Solution

$$\begin{aligned}(x + 4)(x + 7) &= x \cdot x + x \cdot 7 + 4 \cdot x + 4 \cdot 7 \\ &= x^2 + 7x + 4x + 28 \\ &= x^2 + 11x + 28\end{aligned}$$

The first step is usually not written, except as an explanation.

Example 1.20

Expand and simplify $(y + 5)^2$.

Solution

Rewrite $(y + 5)^2$ as $(y + 5)(y + 5)$.

$$(y + 5)(y + 5) = y^2 + 5y + 5y + 25 = y^2 + 10y + 25$$

As with most mathematics, a concept is thought of, a rule made, and a shortcut developed. One shortcut that can be developed (and hence eliminate the middle step of the solution in the example above) is:

$$(a + b)^2 = a^2 + 2ab + b^2$$

Even though the FOIL mnemonic is specific to multiplying two binomials, the idea can be used to expand the product of binomials and trinomials.

Example 1.21

Find and simplify the product $(x + 2)(x^2 + 3x + 5)$.

Solution

$$\begin{aligned}(x + 2)(x^2 + 3x + 5) &= x^3 + 3x^2 + 5x + 2x^2 + 6x + 10\\ &= x^3 + 5x^2 + 11x + 10\end{aligned}$$

Are you able to see the '**FOIL**' method at work here?

Factorizing

A polynomial has been factorized when the answer is in the form of one term. The one term may possibly contain many (two or more) factors.

For example, $x^2 + 7x + 12$ has three terms, but when factorized as $(x + 3)(x + 4)$ it is expressed as only one term (with two factors).

We know that the trinomial has been 'factorized' correctly since when it is expanded', the trinomial, $x^2 + 7x + 12$, reappears.

The concept of factorizing involves expanding. To be able to factorize simple polynomials correctly, you must always expand your (factorized) answer in order to see if the original polynomial has reappeared.

Example 1.22

Factorize $x^2 + 10x + 24$.

Solution

Start by writing two sets of open parentheses: $(\_ + \_)(\_ + \_)$.

Next, fill in the blanks with your best guess so that when you expand the answer you will get $x^2 + 10x + 24$ back again.

Try this guess: $(x + 3)(x + 8)$. This is a good guess since $3 \cdot 8 = 24$.

Now test the guess by expanding:
$(x + 3)(x + 8) = x^2 + 8x + 3x + 24 = x^2 + 11x + 24$

Since this is not $x^2 + 10x + 24$, the factorized form of $(x + 3)(x + 8)$ was not correct. In other words, 3 and 8 were not the correct choices.

Therefore, try another combination of numbers. Try 4 and 6.

Test the new guess by expanding:
$(x+4)(x+6) = x^2 + 6x + 4x + 24 = x^2 + 10x + 24$

Since this is the original polynomial, $x^2 + 10x + 24$ factors correctly as $(x+4)(x+6)$.

Rearranging formulae

Rearranging formulae involves the use of axioms and theorems to transform equations into different but equivalent forms.

For example, the equation $y = 2x + 1$ can be transformed into an equivalent form such as $y - 1 = 2x$. These forms are not equal since the left sides are different, but they are equivalent since they both have the same solution set. In this case the solution set is a set of ordered pairs. A few such ordered pairs would be $(-1, -1)$, $(0, 1)$, and $(1, 3)$.

The strict method for transforming equations requires the use of axioms, definitions and theorems. Although very interesting, those concepts are beyond the scope of this course. There are two practical methods that are often used. The first is used in the beginning of transforming equations and the second is used after proficiency has been gained in the first method.

Method I: Transform $y = 3x - 2$ into the form $ax + by + c = 0$, where $a, b, c \in \mathbb{Z}$.

Step 1: Write the equation: $y = 3x - 2$

Step 2: Thinking of the order of operations in reverse, add 2 to both sides:

$$\begin{array}{r} y = 3x - 2 \\ \underline{+2 \qquad\quad 2} \end{array}$$

Step 3: Simplify the result $\quad y + 2 = 3x$

Step 4: Add $-3x$ to both sides: $\quad y + 2 = 3x$

$$\underline{+-3x \qquad 3x}$$

Step 5: Simplify the result: $\quad -3x + y + 2 = 0$

● **Examiner's hint:** When you are unsure, or the pressure of a test (i.e. classroom, IB, AP, PSAT, SAT, ACT) is intense, Method I is almost foolproof, it is just that it takes so much longer.

Method II: This method involves thinking of the steps in Method 1, not writing them all down, and just simplifying the results.

Step 1: Write the equation: $y = 3x - 2$

Step 2: Add 2 to both sides: $y + 2 = 3x$

Step 3: Add $-3x$ to both sides $-3x + y + 2 = 0$

You can clearly see that Method II is faster since there is less work involved. However, Method II does take a little practice as there are more opportunities for careless mistakes.

The phrase 'y' is in terms of 'x' means that on the left side of the equation is the variable y and on the right side are terms that involve the variable x. In the example $y = 7x + 5$, x does not appear to be present with the constant 5. However, you can think about 5 as $5 \cdot x^0$ and thus 'see' x in all of the terms.

Example 1.23

Write, in words, which variable is in terms of the other(s).

a) $A = \pi r^2$ b) $P = 2l + 2w$

Solution

a) A is in terms of r. b) P is in terms of l and w.

Example 1.24

Given the formula for simple interest, $I = \frac{Crn}{100}$, solve for r in terms of C, n and I.

Solution

$$I = \frac{Crn}{100}$$

$$100\,I = Crn \text{ (both sides multiplied by 100)}$$

$$\frac{100\,I}{Cn} = r \text{ (both sides divided by } Cn\text{)}$$

$$\therefore \quad r = \frac{100I}{Cn}$$

The symbol '$\therefore$' is read as 'therefore'.

Evaluating expressions

To evaluate an expression is to find a number value for the expression.

To evaluate a polynomial expression means to substitute the given value(s) for the variable(s) and then write a simplified answer.

From algebra, it is known that $a^{-b} = \frac{1}{a^b}$, when $b > 0$. For example, $3^{-2} = \frac{1}{3^2}$.

Example 1.25

Evaluate the following:

a) $5 - 2 \cdot 4$ b) $x^2 + 3x + 1$, for $x = 4$ c) 5^{-2}

Solution

a) $5 - 2 \cdot 4 = 5 - 8 = -3$

b) $x^2 + 3x + 1 = 4^2 + 3 \cdot 4 + 1 = 16 + 12 + 1 = 29$

c) $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

Solving linear equations in one variable

To solve an equation means to find an answer that will satisfy the equation.

A linear equation is an equation in which the variable's exponent is 1 and the graph of the related function is a straight line.

See Chapter 4 for a more detailed explanation of linear functions.

Example 1.26

Solve for x.

a) $2x + 3 = 13$ b) $15x - 2(x + 7) = 10x + 19$

Solution

The following solutions will make use of Method II on page 14.

a) $2x + 3 = 13$

$$2x = 13 - 3 = 10$$

$$\therefore \quad x = \frac{10}{2} = 5$$

You should check that 5 is the solution by substituting it back into the original equation and verifying that 5 satisfies the equation.

Does $2 \cdot 5 + 3 = 13$?

Yes, $10 + 3 = 13$.

Therefore, since 5 satisfies the original equation, 5 is the solution.

b)
$$\begin{aligned} 15x - 2(x + 7) &= 10x + 19 \\ 15x - 2x - 14 &= 10x + 19 \\ 13x - 14 &= 10x + 19 \\ 13x - 10x &= 19 + 14 \\ 3x &= 33 \\ \therefore \quad x &= \frac{33}{3} = 11 \end{aligned}$$

As you become more proficient at solving equations, steps such as the 3rd and 4th ones can be skipped.

Example 1.27

Solve the equation $\frac{x}{2} = \frac{3}{5}$.

Solution

Method I:
$$\begin{aligned} 10 \cdot \frac{x}{2} &= 10 \cdot \frac{3}{5} && \text{(multiply both sides by 10, the LCM)} \\ 5x &= 2 \cdot 3 && \text{(simplify)} \\ 5x &= 6 \\ \therefore \quad x &= \frac{6}{5} \end{aligned}$$

Method II:
$$\begin{aligned} 5 \cdot x &= 2 \cdot 3 && \text{(cross-multiply)} \\ 5x &= 6 && \text{(simplify)} \\ \therefore \quad x &= \frac{6}{5} \end{aligned}$$

Method II can often save at least one step.

Exercise 1.3

1. Expand each of the following. Leave your answer as a simplified polynomial.

a) $5(x + 3)$ b) $-3(y - 7)$ c) $x(x + y)$

d) $z(w - t)$ e) $(x + 9)(x + 2)$ f) $(r + 1)(r - 7)$

g) $(2y + 3)(3y - 2)$ h) $(x + 4)(x - 4)$ i) $(2a + 3)^2$

j) $(3z - 1)(3z + 1)$ k) $(x + 3)(x^2 + 2x + 4)$ l) $(g - 5)(g^2 - 7g - 1)$

2. Factorize each polynomial completely over the set of integers.

a) $5x - 5$ b) $3y + 6$ c) $2x^2 + 6x + 8$

d) $5z^2 - 15z + 45$ e) $x^2 + 5x + 6$ f) $y^2 + 8y + 15$

g) $z^2 - z - 2$ h) $w^2 + 4w - 21$ i) $x^2 - 16$
j) $r^2 - 25$ k) $2x^2 - 11x - 21$ l) $3m^2 + 10m + 3$

3. Solve for the underlined variable in terms of the other variable(s).

a) $\underline{\mathbf{y}} - x = 5$ b) $2x + \underline{\mathbf{y}} - 7 = 0$ c) $2\underline{\mathbf{z}} + 4w = 7$

d) $3\underline{\mathbf{r}} - 5s - 6 = 1$ e) $r\underline{\mathbf{t}} = d$ f) $p = \frac{360}{\underline{\mathbf{b}}}$

g) $u = a + (\underline{\mathbf{n}} - 1)d$ h) $u = a + (n - 1)\underline{\mathbf{d}}$ i) $x = \frac{-\underline{\mathbf{b}}}{2a}$

j) $x = \frac{-b}{2\underline{\mathbf{a}}}$ k) $A = \frac{\underline{\mathbf{h}}(a + b)}{2}$ l) $r = \frac{S_{xy}}{S_x \cdot \underline{\mathbf{S_y}}}$

4. Evaluate each expression for the given value of the variable.

a) πr^2; $r = 4$ b) $\frac{4}{3}\pi r^3$; $r = 3$

c) $x^2 + 5x + 1$; $x = 2$ d) $3y^2 - 4y + 5$; $y = -1$

e) $\frac{2}{x - 1}$; $x = 1.1$ f) $\frac{2}{x - 1}$; $x = 1.01$

g) $\frac{2}{x - 1}$; $x = 1.001$ h) $\frac{2}{x - 1}$; $x = 1$

i) $u + (n - 1)d$; $u = 3, n = 25, d = 4$

j) $\frac{u(1 - r^n)}{1 - r}$; $u = 2, r = 0.5, n = 10$

k) $C\left(1 + \frac{r}{100}\right)^n - C$; $C = 1000, r = 6, n = 30$

l) $C\left(1 + \frac{r}{100t}\right)^{nt} - C$; $C = 1000, r = 6, n = 30, t = 12$

m) 7^{-3} n) 2^{-5} o) $\left(\frac{1}{2}\right)^{-1}$ p) $\left(\frac{3}{4}\right)^{-2}$

5. Solve each equation.

a) $2x + 1 = 5$ b) $3y - 7 = 8$

c) $5(z + 1) = 15$ d) $-3(2x - 3) = -9$

e) $2r + 7 = -3r + 27$ f) $-4t - 1 = 2t + 5$

g) $2x + 2 = -(x + 1)$ h) $-(7w - 1) = 7w + 13$

i) $\frac{2}{3}y - 1 = 8$ j) $\frac{-4}{7}t + \frac{3}{14} = \frac{5}{7}$ (Hint: Use Method I)

k) $\frac{x}{5} = \frac{3}{8}$ l) $\frac{-3}{7} = \frac{y}{4}$

m) $\frac{4}{r} = \frac{9}{11}$ n) $\frac{2x + 1}{3} = \frac{5}{7}$

o) $\frac{-3z - 2}{5} = \frac{-5z + 2}{3} + 2$ (Hint: Use Method I)

p) $\frac{-3x + 13}{2} + 1 = \frac{2}{3}$ (Hint: Use Method I)

1.4 Algebra extended

Rewriting linear equations in two variables

Linear equations can be expressed in many forms. Some of these are:

- gradient-intercept
- standard
- point-gradient. (This form is not required at this level.)

Any particular equation can be rewritten in any one of the forms.

Each form is useful in its own way. (See Chapter 3.)

This section will concentrate on rewriting a given linear equation into one of the above forms.

$y = mx + c$ is called the gradient-intercept form of a linear equation.

Example 1.28

Rewrite $2x - 7y = 5$ in the form $y = mx + c$.

Solution

$$\begin{aligned} 2x - 7y &= 5 \\ -7y &= -2x + 5 \\ y &= \frac{-2x + 5}{-7} \\ y &= \frac{-2}{-7}x + \frac{5}{-7} \\ \therefore \quad y &= \frac{2}{7}x + \frac{-5}{7} \end{aligned}$$

You can see that m is $\frac{2}{7}$ and that c is $\frac{-5}{7}$.

An equally good answer would be:

$$y = \frac{2}{7}x - \frac{5}{7}$$

Why is division by zero not allowed? For an explanation, visit www.pearsonhotlinks.co.uk, enter the ISBN for this book and click on weblink 1.4.

$ax + by + d = 0$ is called the standard form of a linear equation.

Example 1.29

Rewrite $y = \frac{2}{3}x - 5$ in the form $ax + by + d = 0$, where $a, b, d \in \mathbb{Z}$.

Solution

$$\begin{aligned} 3 \cdot \left(y = \frac{2}{3}x - 5\right) \\ 3y = 2x - 15 \\ 3y - 2x + 15 = 0 \\ \therefore \quad -2x + 3y + 15 = 0 \end{aligned}$$

You can see that $a = -2$, $b = 3$, and $d = 15$.

An equally good answer would be:

$$2x - 3y - 15 = 0$$

Then, $a = 2$, $b = -3$, and $d = -15$.

Both sides of the equation $-2x + 3y + 15 = 0$ were multiplied by -1.

Solving a system of linear equations in two variables

A system of linear equations contains at least two equations.

Straight line graphs are associated with the equations.

The graphs are to be considered coplanar (lying in the same plane).

The graphs of the equations could:

- intersect at one point
- intersect at all points (both lines would be the same)
- not intersect at all (the lines would be parallel).

Solving a system of equations means finding the point, if one exists, where the graphs intersect.

This review will concentrate on finding the solution for a system of two equations each having the same two variables. This discussion will also assume that the graphs intersect thus producing one unique solution. This solution will be in the form of an ordered pair.

Example 1.30

Solve the system: $2x + 3y = 2$

$$5x + 2y = -6$$

Solution

The Linear Combination Method

Step 1: Multiply both sides of the equation by a number so that, when you add the equations together, the sum of one of the terms will be 0.

- In this case, both sides of the first equation are multiplied by 5, and both sides of the second equation are multiplied by -2.
- The notation $5 \cdot (2x + 3y = 2)$ is really a short form of $5 \cdot (2x + 3y) = 5 \cdot 2$.
- A good hint is to always add instead of subtract. This will reduce careless mistakes.
- The symbol $\Rightarrow$ is read as 'implies'. (See Chapter 9.)

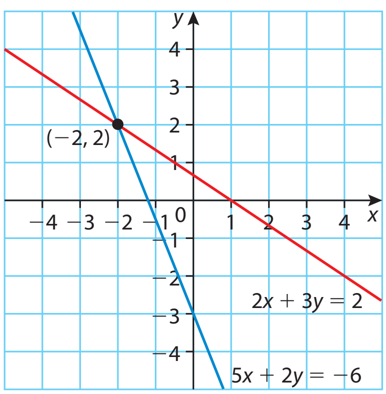
Step 2: Solve the equation that results from the addition.

$$\begin{aligned} 2x + 3y = 2 \quad &\Rightarrow \quad 5 \cdot (2x + 3y = 2) \quad \Rightarrow \quad 10x + 15y = 10 \\ 5x + 2y = -6 \;&\Rightarrow\; -2 \cdot (5x + 2y = -6) \;\Rightarrow\; \underline{-10x - 4y = 12} \\ &\qquad\qquad\qquad\qquad\qquad 0 + 11y = 22 \\ &\qquad\qquad\qquad\qquad\qquad \therefore \quad y = 2 \end{aligned}$$

Step 3: Now substitute $y = 2$ back into any one of the above equations in order to find the x-value. In this case we chose $2x + 3y = 2$, since it looked the easiest.

$$\begin{aligned} 2x + 3 \cdot 2 &= 2 \\ 2x &= -4 \\ \therefore \quad x &= -2 \end{aligned}$$

- Therefore, the solution to the system of equations is the ordered pair $(-2, 2)$. This means that the graphs of the equations will intersect at the ordered pair $(-2, 2)$.



Step 4: It is also a good idea to check this solution by substituting it into the other equation.

$$5 \cdot 2 + 2 \cdot -2 \stackrel{?}{=} -6$$
$$10 + -4 \stackrel{?}{=} -6$$
$$-6 \stackrel{\checkmark}{=} -6$$

Example 1.31

Solve the system:
$$y = 2x + 1$$
$$3x - 5y = 2$$

Solution

The Substitution Method

Step 1: Substitute the right side of the first equation into the y-value in the second equation.

- This method is very useful when one of the equations has one variable in terms of the other variable.

$$3x - 5 \cdot (2x + 1) = 2$$

Step 2: Solve for x:
$$3x - 5 \cdot (2x + 1) = 2$$
$$3x - 10x - 5 = 2$$
$$-7x = 7$$
$$\therefore \quad x = -1$$

Step 3: Back-substitute to find the y-value.

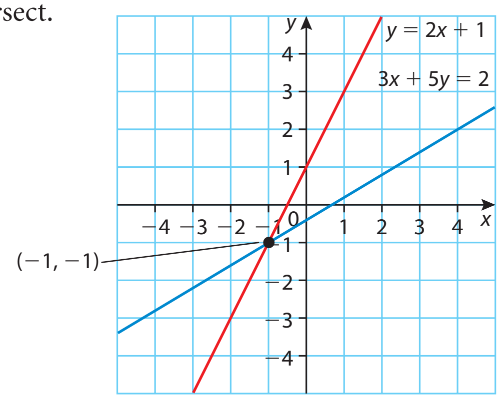
- Although either equation can be used, the first one will be the most efficient since y is already in terms of x.

$$y = 2x + 1$$
$$y = 2 \cdot -1 + 1$$
$$\therefore \quad y = -1$$

Step 4: Check to see if this solution satisfies the other equation.

$$3(-1) - 5(-1) \stackrel{?}{=} 2$$
$$2 \stackrel{\checkmark}{=} 2$$

- Since $2 = 2$, we know that the solution $(-1, -1)$ satisfies both equations and therefore represents the solution of the system.
- As before, this solution represents the ordered pair where the lines intersect.



Order relations

Order relations involve the use of:

- $<$ read as 'less than'
- $\leqslant$ read as 'less than or equal to'
- $>$ read as 'greater than'
- $\geqslant$ read as 'greater than or equal to'.

These relations are often called **inequalities**.

It is advisable, although not necessary, to keep the variable on the left side of the equation.

- For example, $x < 5$ represents the same set of numbers as $5 > x$, but is usually easier for you to graph on a number line when written as $x < 5$.

When graphing on a number line, make the graph extend in the same direction as the inequality symbol is pointing.

- For example, the graph of $x < 5$ would point like this: 
- The graph of $x \geqslant 7$ would point like this:
- When graphing using $<$ or $>$ indicate the end of the graph with an open circle. This tells the reader that you are not including the endpoint.
- When using $\leqslant$ or $\geqslant$ indicate the end of the graph with a closed circle. This tells the reader that you are including the endpoint.

The rules for the use of order relations when solving inequalities are:

- the same for addition and subtraction, and you will *keep the same* inequality symbol when performing those operations.
- the same for multiplication and division *except* that you will *reverse* the inequality symbol you are using when performing those operations with *negative* numbers.

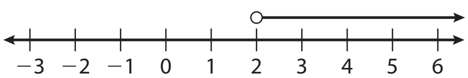
Example 1.32

Solve and graph the solution on a number line: $2x + 1 > 5$

Solution

$$\begin{aligned} 2x + 1 &> 5 \\ 2x &> 5 - 1 \\ 2x &> 4 \\ x &> \frac{4}{2} \\ \therefore \quad x &> 2 \end{aligned}$$

Notice that the inequality symbol stayed the same when subtracting.



Notice that the inequality symbol stayed the same when dividing by 2.

Example 1.33

Solve and graph on a number line: $3x + 7 \geqslant 8x + 27$

Solution

$3x + 7 \geqslant 8x + 27$

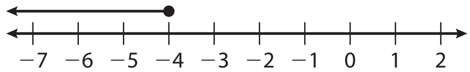
$3x - 8x \geqslant 27 - 7$ Hint: Notice that the inequality symbol stayed the same when we subtracted.

$-5x \geqslant 20$ Hint: The inequality symbol is <u>still</u> the same.

$$x \leqslant \frac{20}{-5}$$

Hint: When we divided by -5, the inequality symbol reversed!

$$\therefore \quad x \leqslant -4$$



Intervals on a real number line

$\{x|x > 5, x \in \mathbb{R}\}$ is read as, 'the set of all x such that x is greater than 5 and x is an element of the real numbers.'

The real number line is completely filled up with the rational ($\mathbb{Q}$) and irrational ($\mathbb{Q}'$) numbers.

There are no gaps on the real number line. Every point has one and only one real number assigned to it and conversely, every real number has one and only one point assigned to it.

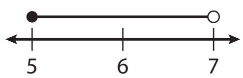
There are several ways to express a group of real numbers you wish to discuss. For example, if you wish to discuss the group of real numbers greater than 5, you could write that idea in any one of the following ways:

The set of real numbers would be implied when writing $\{x|x > 5\}$.

- $x > 5$
- $\{x|x > 5, x \in \mathbb{R}\}$
- $\{x|x > 5\}$
- $(5, \infty)$

The last notation is called **interval notation** and is explained below.

- A parenthesis, (or), is an indication not to include the real number next to it. In $(5, \infty)$, the parenthesis next to the 5 would mean that the group of numbers being discussed would not include 5, but would include any real number to the right of 5 on the number line, i.e. 5.1, 5.01, 5.001, 5.0001, etc.
- The infinity symbol, ∞, will always have a parenthesis next to it as it is assumed that you can never reach infinity.
- A bracket, [or], would indicate to the reader that you mean to include the number next to it in the group of numbers you wish to discuss. For example, $[5, 7)$, would be the group (the set) of real numbers from 5 to 7. The number 5 would be included in the set, but 7 would not be. The graph would look like:



A word of caution – the interval (2, 8) can be confused with the ordered pair (2, 8). It is therefore important to keep the context of the question in mind at all times.

Example 1.34

Write each of the following inequalities in interval notation.

a) $x > 7$ b) $-3 \leqslant x < 9$

Solution

a) $(7, \infty)$ b) $[-3, 9)$

Exercise 1.4

1. Write each of the following in the form $y = mx + c$, where $m, c \in \mathbb{Q}$.
   a) $2x + y = 9$ b) $3x - 2y = 7$ c) $\frac{3}{2}x + \frac{5}{2}y = 4$
   d) $6x - \frac{4}{7}y = 1$ e) $0.2x + 2.8y = 0.5$

2. Write each of the following in the form $ax + by + d = 0$, where $a, b, d \in \mathbb{Z}$.
   a) $\frac{2}{3}x - \frac{5}{6}y = 1$ b) $y = 4x - 5$ c) $0.5x + 1.2y = 0.7$ d) $y = \frac{-3}{7}x + \frac{2}{7}$

3. Write each of the following in the form $ax + by + d = 0$, where $a, b, d \in \mathbb{Z}$, $a > 0$.
   a) $y = \frac{1}{2}x - \frac{3}{4}$ b) $y = 0.3x + 0.5$

4. Solve each system using the Linear Combination Method. Leave your answer in exact form.
   a) $2x + 3y = 12$
   $5x - 2y = 11$
   b) $4x - 7y = 3$
   $3x - 5y = 1$
   c) $5x - 6y = 2$
   $-4x + 2y = -3$
   d) $-9x - 7y = 8$
   $2x + 6y = 11$

5. Solve each system using the Substitution Method. Leave your answer in exact form.
   a) $y = 2x + 1$
   $3x + y = 11$
   b) $x + y = 2$
   $3x + 2y = 1$
   c) $y = 5x - 1$
   $y = -6x + 7$
   d) $y = \frac{4}{5}x + 1$
   $y = \frac{2}{3}x - 6$

6. Solve each inequality. Graph the solution set on a number line.
   a) $2x - 1 > 9$ b) $-3z - 2 \geqslant 19$
   c) $5(t - 3) + 1 < 2t + 4$ d) $7 \leqslant -(r - 1)$
   e) $\frac{-6}{7}m + 1 \leqslant \frac{3}{14}m + 2$ f) $8 > 2w - 10$

7. Write the following inequalities in interval notation.
   a) $x > 7$ b) $y \leqslant 4$ c) $-5 \leqslant z \leqslant 6$
   d) $13 < t \leqslant 25$ e) $r \geqslant -3$ f) $8 < x < 12$

8. Write the following interval notations in inequality form.
   a) $[2, \infty)$ b) $(-\infty, 9)$ c) $[-2, 8)$
   d) $(3, 10)$ e) $(-\infty, \infty)$ f) $(-6, -1]$
   g) $[3, 4]$ h) $(-5, 0)$

1.5 Geometry

Basic concepts

In any mathematical system there must be a starting place or beginning position. This place is called the **undefined terms**. Undefined terms are those 'things' we believe exist but we just cannot write a definition for. For example, 'addition' $(+)$ is an undefined term until we give meaning to it with the addition tables. In geometry there are three undefined terms.

- Point: it has no length, width, or depth. It has no dimensions. For example, think of a pencil dot on a piece of paper.

How do these terms relate to the ideas expressed in Euclid's *Elements*?

How is it that we can discuss something that we cannot see or touch? If we can think about something, then must that something exist?

- Line: it has no width or depth. It has only length. A line extends infinitely far in one dimension. For example, think of a red laser beam shot through a smoke-filled room.
- Plane: it has no depth. It has length and width only. A plane extends infinitely far in two dimensions. For example, think of the smooth surface of a lake.

The above examples are only useful to the extent that they are trying to convey what we all already believe is true. The three undefined terms are only concepts we can think of mentally. It does seem, however, that we intuitively know what they mean.

Once the three undefined terms have been established, then the study of geometry can proceed. The next ideas that need to be discussed are **definitions**. An example of a geometric definition is **space**.

Space is defined as the collection of all points. Space has length, width and depth. The best example of space is the universe.

Once the undefined terms and some definitions have been established, then **postulates** may be introduced.

For some of the proofs of Pythagoras' theorem, visit www.pearsonhotlinks.co.uk, enter the ISBN for this book and click on weblinks 1.5 and 1.6.

A postulate is a statement of fact in which we have complete faith, but are unable to prove. For example, the statement 'every two points will contain one and only one line' is considered a postulate since it agrees fundamentally with our common and intuitive senses. However, we cannot prove it. We simply accept it as fact. We believe it is true. We have faith it is true.

Finally, after the undefined terms, some defined terms, and some postulates have been introduced, then **theorems** can be hypothesized.
A theorem is a statement of fact that can be proven true based upon the undefined terms, defined terms and the postulates.

- One of the best known theorems is Pythagoras' theorem.
- Another one is: The sum of the measures of the angles of a triangle is 180 degrees.

For a more complete discussion on the basic concepts of geometry, visit www.pearsonhotlinks.co.uk, enter the ISBN for this book and click on weblink 1.7.

Exercise 1.5A

1. List three examples you might use to describe a point.
2. List three examples you might use to describe a line (or line segment).
3. List three examples you might use to describe a plane (or piece of a plane).
4. How does 'circular reasoning' differ from 'logical reasoning'?
5. Why is subtraction not an undefined term?
6. List three examples of geometry definitions.
7. How do postulates differ from undefined terms?
8. List three examples of geometry postulates.
9. How do theorems differ from postulates?
10. List three examples of geometry theorems.

Perimeter and areas of two-dimensional shapes

The **perimeter** of a closed shape is defined as the distance around that shape.

If the closed shape is a polygon, the perimeter is the sum of the lengths of its sides.

If the closed shape is a circle, the perimeter is called the circumference.

Below are some of the formulae used to compute perimeter.

- Perimeter of a rectangle: $P = 2l + 2w$, where l is the length and w is the width.
- Perimeter of a square: $P = 4s$, where s is the length of a side.
- Perimeter of a triangle: $P = a + b + c$, where a, b and c are the lengths of the sides.
- Perimeter of a circle: $C = 2\pi r$, where C is the circumference and r is the radius.

The **area** of a closed shape is defined as the number of square units it contains.

In layman's terms, area is the size of the surface of a closed figure.

In some geometry books, an area postulate is used prior to defining area.

Below are some of the many formulae used to compute areas.

- Area of a rectangle: $A = (l \times w)$, where l is the length and w is the width.
- Area of a parallelogram: $A = (b \times h)$, where b is the base and h is the height.
- Area of a triangle: $A = \frac{1}{2}(b \times h)$, where b is the base and h is the height.
- Area of a trapezium: $A = \frac{1}{2}(a + b)h$, where a and b are the parallel sides and h is the height.
- Area of a circle: $A = \pi r^2$, where r is the radius.

See Chapter 2 for a more detailed explanation of SI units.

In most countries the SI (Le Système International) units for length and area are used. The unit for length is metre (m) and area is square metre (m^2). The metre is made up of smaller sub units according to a decimal system.

1 m = 100 centimetres (cm)
1 cm = 10 millimetres (mm)

When calculating areas we need to be a little careful.

1 square metre (m^2) = 1 m × 1 m = 100 cm × 100 cm = 10 000 cm^2
1 square centimetre (cm^2) = 10 mm × 10 mm = 100 mm^2

Example 1.35

Find the perimeter and area of a rectangle whose dimensions are 45 cm long and 30 cm wide.

Solution

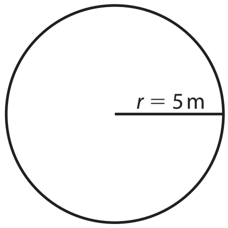
Perimeter: $P = 2(l + w) = 2(45\text{ cm} + 30\text{ cm}) = 2(75\text{ cm}) = 150\text{ cm}$

Area: $A = l \times w = 45\text{ cm} \times 30\text{ cm} = 1350$ square cm $= 1350\text{ cm}^2$

Example 1.36

Find the perimeter and area for the given circle

a) exactly b) to 3 significant figures.



Solution

Perimeter: $C = 2\pi r = 2\pi \times 5 = 10\pi$ m. This is the exact answer.

$C = 2\pi r = 2 \times 3.14159 \times 5 = 31.4159 = 31.4$ m to 3 s.f.

This is the approximate answer.

Area: $A = \pi r^2 = \pi \times 5^2 = \pi \times 25 = 25\pi$ square metres $= 25\pi\text{ m}^2$.

This is the exact answer.

$A = \pi r^2 = 3.14159 \times 5^2 = 78.53975 = 78.5\text{ m}^2$ to 3 s.f.

This is the approximate answer.

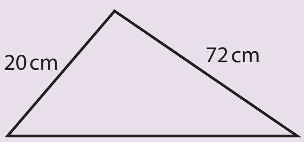
● **Examiner's hint:** It is commonly accepted only to use the unit of measure, cm, or cm^2, with the final answer.

Exercise 1.5B

1. For each of the following diagrams find the perimeter. Express the answer exactly.

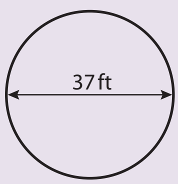
a) Parallelogram:

b) Triangle:

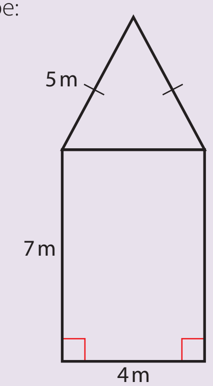


c) Trapezium:

d) Circle:



e) Compound shape:



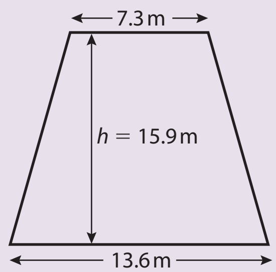
2. Find each of the following:

a) The exact area of a rectangle whose dimensions are 100 m by 50 m.

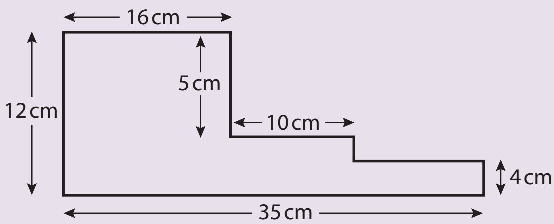
b) The exact area of a triangle whose base is 22.5 cm and height is 10.4 cm.

c) The area of a circle whose radius is 17 feet, giving your answer correct to 3 s.f.

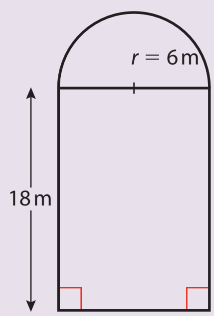
d) The area of the trapezium below, giving your answer correct to 1 decimal place.



e) The exact area of the compound shape below.



f) The approximate area of the compound shape below correct to 1 decimal place.



How is it possible to think of a line, which is an undefined term, in terms of points which have no dimensions?

Are there more points on the line or on the plane? How does this help us understand the universe in which we live?

Plotting on the *x*-, *y*-coordinate plane

Think of a line as infinitely many points crammed together in a straight row so that there is no space between them.

Think of a plane as infinitely many points crammed together on a flat surface so that there is no space between them.

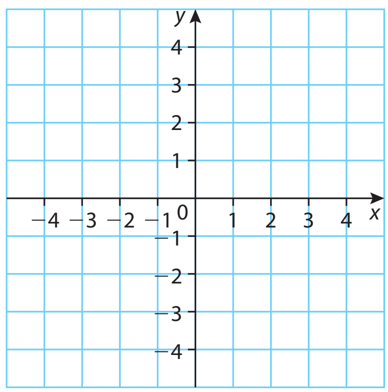
The word 'coordinate' is derived from the prefix 'co' meaning 'together with' and 'ordinal' meaning 'of a certain order'.

Therefore, 'coordinate' means 'a set of numbers in a specified order'.

Hence, a coordinate system is a way to locate a point either on a line, in a plane or in space.

In a coordinate plane only two numbers are required to locate a point.

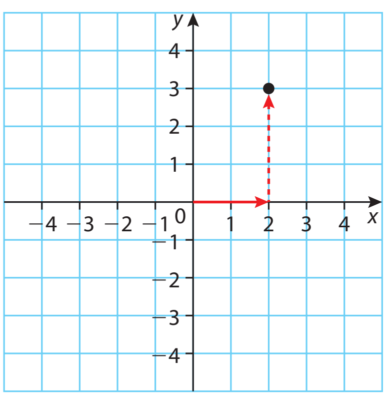
- The first number is called the **abscissa**, often referred to as x. The second is called the **ordinate**, often referred to as y.
- Such a system is often referred to as an x-, y-coordinate plane.
- The axes are often labelled x for the horizontal movement and y for the vertical movement. They meet at the origin (0, 0).
- To the left and below (0, 0) are the negative numbers and to the right and above are the positive numbers.
- An x-, y-coordinate plane looks like this:



- To locate a point in the plane start at the origin and move horizontally (right or left) and then vertically (up or down) according to the ordered pair given.

Example 1.37

Locate and plot (2, 3) on the x-, y-coordinate plane.



Solution

Start at the origin.
Move 2 units to the right.
Move up 3 units parallel to the y-axis.

Exercise 1.5C

On the same x-, y-coordinate axes locate and plot each coordinate.

1. (3, 4)	**2.** (−3, 2)	**3.** (−4, −5)	**4.** (1, 1)	**5.** (0, 3)
6. (−2, 0)	**7.** (3, −3)	**8.** (−1, 5)	**9.** (−2, 1)	**10.** (5, −1)

Midpoint of a line segment

The midpoint of a line segment is defined to be the point, expressed in x and y coordinates, where a line segment is broken into two equal pieces. Simply put, it is the point where the middle of a line segment occurs.

To find this point, given the coordinates of the endpoints of the line segments, we find the mean of the x component of each coordinate and the mean of the y component of each coordinate. Therefore, the formula for the midpoint of a line is:

$$M(x_M, y_M) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

For a helpful applet that shows the midpoint of a line, visit www.pearsonhotlinks.co.uk, enter the ISBN for this book and click on weblink 1.8

Example 1.38

Given $A(2, 3)$ and $B(-6, 0)$, find the midpoint of A and B.

Solution

We know that $M(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$, therefore:

$$\begin{aligned} M(x, y) &= \left(\frac{2 + -6}{2}, \frac{3 + 0}{2}\right) \\ &= \left(\frac{-4}{2}, \frac{3}{2}\right) \\ &= (-2, 1.5) \end{aligned}$$

Examiner's hint: Always write the formula you intend to use to solve a problem. Some of the marks for a problem come from your method, which you can demonstrate with the use of the proper formula.

Occasionally we will be required to work out one of the endpoints of the line segments, given the midpoint of the line and one of the other endpoints. To do this, we use some relatively straightforward algebra.

Example 1.39

Find the coordinates of B, given that A is $(4, 5)$ and the midpoint of $[AB]$ is $M(1, 3)$.

Solution

First we write down the formula, $M(x_M, y_M) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$, and substitute the values from the question.

$$M(1, 3) = \left(\frac{4 + x_2}{2}, \frac{5 + y_2}{2}\right)$$

This leads to a pair of two-step algebra problems which we need to solve.

$$1 = \frac{4 + x_2}{2} \text{ and } 3 = \frac{5 + y_2}{2}$$

We solve each of these equations in a similar way. We multiply by 2, and isolate the variable using subtraction.

$$\begin{aligned} 2 &= 4 + x_2 \text{ and } 6 = 5 + y_2 \\ -2 &= x_2 \text{ and } 1 = y_2 \end{aligned}$$

Why do you think the formula for the midpoint of a line segment involves finding means?

1. Midpoint formula:
$M(x_M, y_M) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
2. The midpoint of a line segment is the centre of the line segment.

From the solutions to these two equations, we see that the coordinates of B are $(-2, 1)$. We can use the procedure shown in the first example to verify this fact.

Exercise 1.5D

1. Find the midpoint of each line segment.

a) $A(2, 3)$ and $B(5, 6)$

b) $A(-4, -4)$ and $B(4, 4)$

2. Find the midpoint of each side of triangle ABC.

$A(-3, -2)$, $B(3, 3)$, and $C(6, -2)$

3. Draw the triangle using the coordinates $A(-4, 2)$, $B(4, 4)$, and $C(0, -5)$. Find the midpoint of each side of the triangle, then join the midpoints together to form another triangle.

4. Given A and M, the midpoint of $[AB]$, find B.

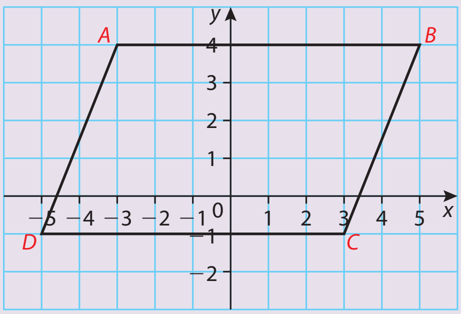
$A(-2, -2)$ and $M(-2, 3)$

5. A has coordinates $(6, -4)$ and B has coordinates $(-4, 2)$.

a) Find the coordinates of M, the midpoint of $[AB]$.

b) Find the coordinates of the midpoint of $[BM]$.

6. Show that the midpoint of $[AC]$ is the same as the midpoint of $[DB]$ in $ABCD$.



7. Redha throws a ball from the corner of a field. When the ball is halfway to where he estimates it will land, he notices that the ball is exactly 40 m north and 30 m west of where he stands. Where will the ball land, relative to Redha?

Right triangles and Pythagoras' theorem

Right triangles

A right triangle is a triangle that contains a 90 degree angle. See Figure 1.3 on page 31.

A, B, C are called the vertices of the triangle.

• **Examiner's hint:** $B\hat{A}C$ and $C\hat{A}B$ are often used by the IBO examiners.

The angles of a triangle can be named in several different ways. For example, Angle A can be named as $\angle A$, or $\measuredangle A$, or $\sphericalangle A$, or $\angle BAC$, or $\measuredangle CAB$, or $B\hat{A}C$, or $C\hat{A}B$.

The sum of the measures of all three angles of a triangle is 180°.

Angle C is called the right angle.

A right angle has a measure of 90 degrees (90°).

Side AB can be named as $\overline{AB}$ or $[AB]$.

The length of $[AB]$ can be named as c or AB.

The hypotenuse is the side opposite the right angle. In this triangle $[AB]$ is the hypotenuse.

The sides that form the right angle are called the legs of the triangle. In this triangle, the legs are $[BC]$ and $[AC]$ while their lengths are a and b respectively.

● **Examiner's hint:** $[AB]$ is the common usage for the IBO examiners.

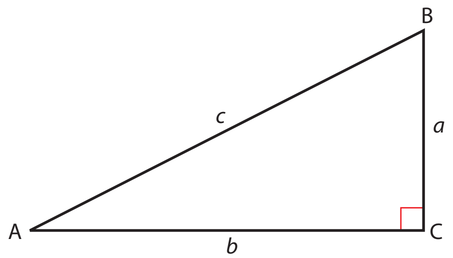


Figure 1.3 Right triangle.

Pythagoras' theorem

Pythagoras, who was born around 575 BC, is credited with discovering and proving arguably the most important theorem in mathematics: Pythagoras' theorem.

The theorem states that if $\triangle ABC$ is a right triangle, then $a^2 + b^2 = c^2$, where a and b are the lengths of the legs and c is the length of the hypotenuse. (See diagram above.)

For example, if $\triangle ABC$ is a right triangle then the sides could have length $a = 3$, $b = 4$, and $c = 5$, since $3^2 + 4^2 = 5^2$.

3, 4, 5 is called a Pythagorean triple.

Other popular Pythagorean triples are: (6, 8, 10), (5, 12, 13), (9, 12, 15), (8, 15, 17).

The converse (see Chapter 9) of Pythagoras also holds true: if a, b, and c are the lengths of the sides of the triangle and $a^2 + b^2 = c^2$, then the triangle is a right triangle.

W For a more complete history of Pythagoras, visit www.pearsonhotlinks.co.uk, enter the ISBN for this book and click on weblink 1.9.

i Not all converses hold true. Chapter 9 explains this further.

Example 1.40

If $\triangle DEF$ is a right triangle, which of the following sets could be the lengths of the sides?

a) 7, 24, 26 b) 9, 40, 41 c) 12, 36, 35

d) 66, 63, 16 e) 28, 53, 45

Solution

a) $7^2 + 24^2 = 625$; $26^2 = 676$. $\therefore$ 7, 24, 26 cannot be the lengths of the sides of $\triangle DEF$.

b) $9^2 + 40^2 = 1681$; $41^2 = 1681$. $\therefore$ 9, 40, 41 *could* be the lengths of the sides of $\triangle DEF$.

c) $12^2 + 35^2 = 1369$; $36^2 = 1296$. $\therefore$ 12, 36, 35 cannot be the lengths of the sides of $\triangle DEF$.

d) $16^2 + 63^2 = 4225$; $66^2 = 4356$. $\therefore$ 66, 63, 16 cannot be the lengths of the sides of $\triangle DEF$.

e) $28^2 + 45^2 = 2809$; $53^2 = 2809$. $\therefore$ 28, 53, 45 *could* be the lengths of the sides of $\triangle DEF$.

Example 1.41

If $\triangle ABC$ is a right triangle with C at the vertex of the right angle and $a = 3$, $b = 7$, solve for c.

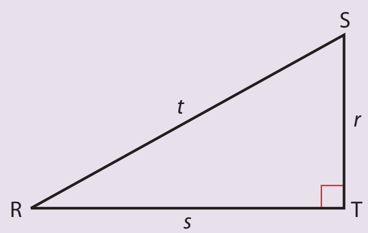
Solution

Since $\triangle ABC$ is a right triangle with C at the vertex of the right angle, then we know that $a^2 + b^2 = c^2$. Therefore, by substitution:

$$\begin{aligned} 3^2 + 7^2 &= c^2 \\ 9 + 49 &= c^2 \\ 58 &= c^2 \\ \therefore \quad c &= \sqrt{58} \end{aligned}$$

Exercise 1.5E

1. Given $\triangle RST$ below, name each of the following:



a) The hypotenuse

b) The right angle (use three different letter designations)

c) The two legs

d) $\angle R + \angle S$

e) Two expressions for the length of $[RS]$

f) Two expressions for the length of $[ST]$

g) Two expressions for the length of $[RT]$

2. Which of the following could be the lengths of the sides of a right-angled triangle?

a) 65, 72, 97 b) 21, 20, 27 c) $1, \sqrt{2}, 1$ d) 1, 1, 1.414

e) $1, 2, \sqrt{3}$ f) 6.5, 42, 42.5 g) $\frac{39}{7}, \frac{89}{7}, \frac{80}{7}$

3. List all of the following that are Pythagorean triples.

a) 12, 35, 37 b) $\frac{1}{2}, 1, \frac{\sqrt{3}}{2}$ c) 10, 10, 20 d) 41, 9, 40

e) 8.100, 15.200, 17.223 f) 1.5, 2, 2.5 g) 55, 48, 73

4. Using $\triangle RST$ above, solve each of the following as required.
   a) If $r = 6$ and $s = 8$, find t (exactly).
   b) If $r = 7$ and $s = 11$, find t (exactly).
   c) If $r = 8$ and $s = 12$, find t (correct to 1 decimal place).
   d) If $r = 23$ and $s = 35$, find t (correct to 1 decimal place).
   e) If $r = 10$ and $t = 26$, find s (exactly).
   f) If $s = 17$ and $t = 30$, find r (exactly).
   g) If $s = \frac{13}{5}$ and $t = \frac{65}{7}$, find r (correct to 1 decimal place).

Pythagoras created a school in Crete for mathematicians to exchange ideas. However, when he disagreed with a particularly controversial result of one of his disciples Pythagoras had him drowned!

Distance between two points

The distance between two points A and B is equal to the length of the line segment joining A to B (see Figure 1.4).

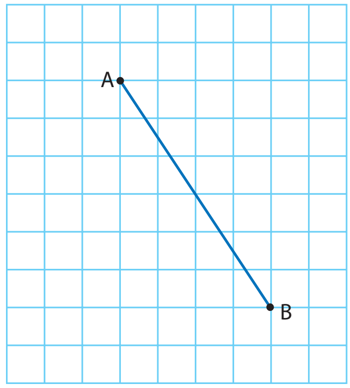


Figure 1.4 The line segment *AB*.

To find the distance between two points on the coordinate plane, there are two widely accepted methods. One is to use the distance formula $\left(d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right)$ which is given in the formula sheet.The other method is to draw a right-angled triangle using the two coordinates, and apply Pythagoras' theorem. Both of these methods involve the same calculations, and both are acceptable ways of finding the distance between two points.

To use the distance formula for the points $A(x_1, y_1)$ and $B(x_2, y_2)$, simply substitute the values of x_1, y_1, x_2 and y_2 into the formula, and then evaluate the formula to find the value of d.

Can you prove that the Pythagorean theorem and the distance formula are equivalent methods for finding the distance between two points?

Example 1.42

Find the distance between $A(1, 2)$ and $B(4, -5)$ to the nearest tenth.

Solution

Use the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ and substitute.

$$\begin{aligned} d &= \sqrt{(4-1)^2 + (-5-2)^2} \\ &= \sqrt{(3)^2 + (-7)^2} \\ &= \sqrt{9 + 49} \\ &= \sqrt{58} \\ d &\approx 7.6 \end{aligned}$$

For a helpful applet that demonstrates the effect of changing the location of the points when using the distance formula, visit www.pearsonhotlinks.co.uk, enter the ISBN for this book and click on weblink 1.10.

The other method is more useful if the problem given has the two points shown in a plot, or if the problem provides a space to plot the two points carefully (like square paper). Once we have the two points plotted, we can create a right-angled triangle, as shown in Figure 1.5. With this right-angled triangle, we can count how many units long the two sides adjacent to the right angle are, and use these measurements in Pythagoras' theorem.

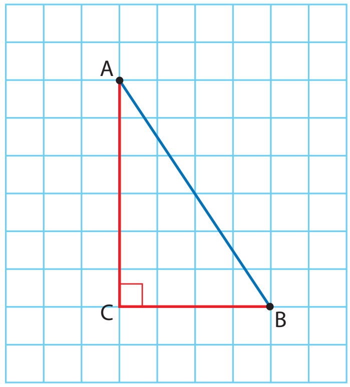
Despite the fact that we attribute the theorem that bears his name to Pythagoras, the same theorem was in use in what is now modern day Iran and in China at least 500 years before he started using it.

How do we measure the distance between two points on Earth?

Example 1.43

In Figure 1.5, how long is [*AB*]?

Figure 1.5 Creating a right-angled triangle from two points.



Solution

The height of the triangle shown is 6 units and the base of the triangle is 4 units.

Use Pythagoras' theorem, $a^2 + b^2 = c^2$, to solve.

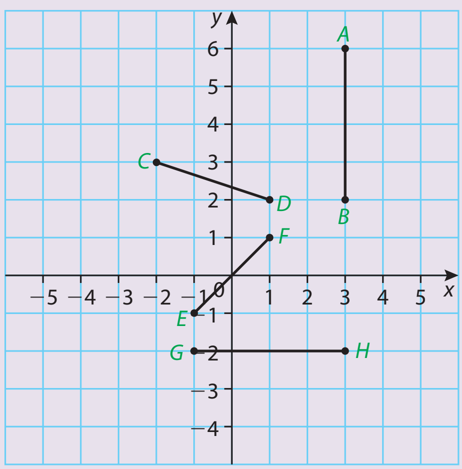
$$6^2 + 4^2 = c^2$$
$$36 + 24 = c^2$$
$$60 = c^2$$
$$\sqrt{60} = c \text{ or } c \approx 7.75$$

Distance formula:
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
The distance formula is equivalent to the Pythagorean theorem.

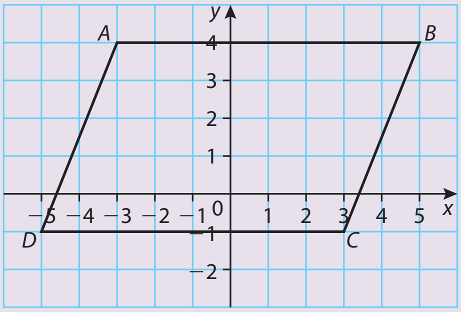
Exercise 1.5F

1. Find the distance between each pair of points.
   a) $A(1, 3)$ and $B(1, 8)$
   b) $A(0, 0)$ and $B(3, 3)$
   c) $A(-1, -1)$ and $B(-4, -5)$

2. Find the lengths of the line segments in the diagram below.

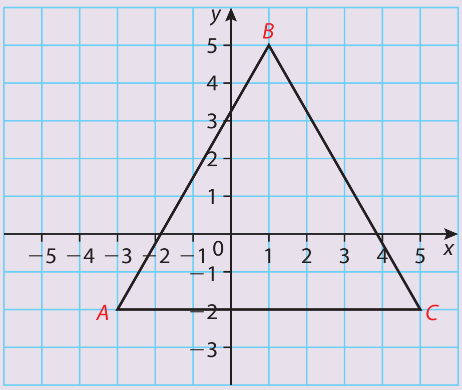


3. Use the distance formula to confirm that $ABCD$ is a parallelogram.



4. A taxi driver is 4 blocks north and 3 blocks east of Union Square in Manhattan, NY. His next customer is located 2 blocks south and 1 block west of Union Square.
   a) Calculate the straight-line distance the taxi driver is from his next customer.
   b) If the taxi driver can only travel south and east, how many blocks does he need to travel to pick up his customer?
   c) How much distance would he save if he could travel straight to his customer?

5. a) Find the length of each side of the triangle ABC.
   b) Round each side length to the nearest whole number.



1.6 Financial mathematics

This section will review the names and abbreviations for common world currencies. We will also review two ways to convert between currencies.

World currencies

The following are examples of some currencies and their abbreviations:

Country	Currency	Abbreviation
United Kingdom	pound	GBP or GB£
United States	dollar	USD or US$
France	euro	EUR or €
South Africa	rand	ZAR
Japan	yen	JPY or ¥
Australia	dollar	AUD or AU$
Spain	euro	EUR or €

W There are a total of about 190 countries, and 167 currencies in use in the world today. For a complete list of all of the world currencies, visit www.pearsonhotlinks.co.uk, enter the ISBN for this book and click on weblink 1.11.

W To access a table listing many of the current exchange rates between currencies, visit www.pearsonhotlinks.co.uk, enter the ISBN for this book and click on weblink 1.12.

When currencies are exchanged (euro for Japanese yen, for example), there is a fee called a commission for doing the exchange. This concept will be discussed in Chapter 2.

Currency conversion

Example 1.44

If 1 USD = 0.501 013 GBP, find

a) the number of GBP for 1250 USD

b) the number of USD for 750 GBP.

Solution

a) Method I:

$$1\text{ USD} = 0.501\,013\text{ GBP}$$
$$1250 \cdot (1\text{ USD}) = 1250 \cdot (0.501\,013\text{ GBP})$$
$$\therefore \quad 1250\text{ USD} = 626.266\,25\text{ GBP} = 626.27\text{ GBP to 2 decimal places.}$$

Make sure you use words when setting up proportion so that you can see 'like' words are in the numerators and 'like' words are in the denominators.

Method II: When using this method, it is important to think of the equation 1 USD = 0.501 013 GBP as the proportion

$$\frac{1\text{ USD}}{0.501\,013\text{ GBP}} \text{ or as } \frac{0.501\,013\text{ GBP}}{1\text{ USD}}.$$

$$\frac{1\text{ USD}}{0.501\,013\text{ GBP}} = \frac{1250\text{ USD}}{x\text{ GPB}}$$

$$(1\text{ USD})(x\text{ GBP}) = (0.501\,013\text{ GBP})(1250\text{ USD})$$

$$\therefore 1 \cdot x = 0.501\,013 \cdot 1250 = 626.266\,25 = 626.27 \text{ to 2 decimal places.}$$

Think of USD and GBP as variables and 'cancel' them out.

b) Method I:

$$1\text{ USD} = 0.501\,013\text{ GBP}$$
$$\frac{1\text{ USD}}{0.501\,013} = \frac{0.501\,013\text{ GBP}}{0.501\,013}$$
$$\therefore \quad 1\text{ GBP} = 1.995\,956\text{ USD to 7 s.f.}$$

Hence, $750(1\text{ GBP}) = 750(1.995\,956\text{ USD})$

and $750\text{ GBP} = 1496.97\text{ USD to 2 decimal places.}$

Method II:

$$\frac{x\text{ USD}}{750\text{ GBP}} = \frac{1\text{ USD}}{0.501\,013\text{ GBP}}$$
$$(x\text{ USD})(0.501\,013\text{ GBP}) = (1\text{ USD})(750\text{ GBP})$$
$$0.501\,013\,x = 750$$
$$x = \frac{750}{0.501\,013}$$
$$\therefore \quad x = 1496.97 \text{ to 2 d.p.}$$

Exercise 1.6

1. Using a website, find the currency and ISO abbreviation for each of the countries.

a) Austria b) Denmark c) China d) Israel

e) Mexico f) Saudi Arabia g) Taiwan

2. Using the website link 1.12, find the currency and the most recent exchange rate (per United States dollar) for each country.

   a) Belgium b) Botswana c) Hong Kong d) United Kingdom
   e) Norway f) Russia g) Switzerland

3. If 1 USD = 10.810 65 MXN, then using Method I, convert:

   a) 225 United States dollars to Mexican pesos correct to 2 decimal places
   b) 350 Mexican pesos to United States dollars correct to 2 decimal places.

4. If I USD = 1.300 540 NZD, then using Method II, convert:

   a) 575 United States dollars to New Zealand dollars correct to 2 decimal places
   b) 2000 New Zealand dollars to United States dollars correct to 2 decimal places.

5. If 1 NOK = 20.819 041 JPY, then using Method I, convert:

   a) 300 Norway krone to Japanese yen correct to the nearest yen.
   b) 16 450 Japanese yen to Norway krone correct to the nearest krone.

6. If 1 EUR = 35.9852 RUB, then using Method II, convert:

   a) 1000 Russian rubles to euros correct to the nearest euro.
   b) 1000 euros to Russian rubles correct to the nearest ruble.

1.7 Statistics

Statistics is largely concerned with the collection of data and the subsequent analysis of that data. A more thorough discussion will be presented in Chapters 11 and 12. This section will merely review some of the very basic concepts: data collection and visual data representation in the forms of bar charts, pie charts and pictograms.

Data collection

Data is information that is used for calculating or measuring. For example:

- A baseball manager might record how many throws a pitcher has made.
- A vehicle manufacturer might record how many hours it takes to make a car.
- A student might record all of her exam grades.

Data can be recorded in several ways. For example:

- Using 'tally marks': e.g. ~~////~~ // might indicate seven shots taken on goal during a football game.
- Using the actual data: e.g. 12.1, 12.7, 13.2, 12.5 might indicate the number of seconds that four people ran the 100 metre dash.

There are three commonly used ways that data can be classified.

- Nominal: no order or ranking can be used with this type of data. For example:
  - political affiliation
  - religious preference.

- Ordinal: an order or ranking can be used, but specific differences between the rankings cannot be determined. For example:
  - letter grades A, B, C, D, F
  - 1st, 2nd, or 3rd place.
- Interval: an order or ranking can be used and specific differences can be determined. For example:
  - test scores
  - height.

Bar charts

Bar charts have the following characteristics:

- The frequency is displayed on the vertical axis.
- The description of each item of data is given below the horizontal axis.
- The bars may not touch.
- The bars have uniform width.
- The bars are often coloured or shaded for visual appeal.

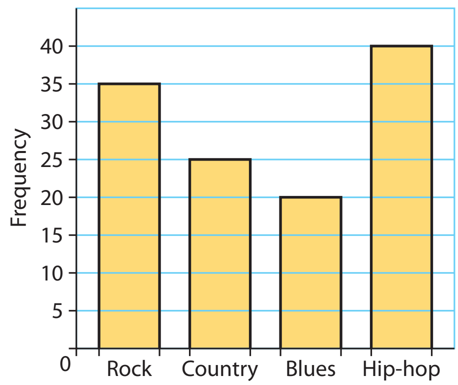
Examiner's hint: Visual displays, such as bar charts, are encouraged when writing the Mathematical Studies project.

Example 1.45

Data has been collected from 120 teenage girls on the type of music they enjoy. The data is shown in the table below. Construct a bar chart to represent the data.

Rock	Country	Blues	Hip-hop
35	25	20	40

Solution



Pie charts

A pie chart is a graph in the shape of a circle. It has the following characteristics:

- It is divided into pie-shaped or wedge-shaped sections.
- Each section represents a percentage of the total amount of data collected.
- The size of each wedge is in terms of degrees and is a percentage of 360°.

- The formula for determining each wedge size is:

$$\text{Wedge size (in degrees)} = \frac{\text{number of data pieces per category}}{\text{total amount of data collected}} \times 360°.$$

Example 1.46

Determine each wedge size for the data given in Example 1.45.

Solution

Rock wedge: degrees $= \frac{35}{120} \cdot 360° = 105°$

Country wedge: degrees $= \frac{25}{120} \cdot 360° = 75°$

Blues wedge: degrees $= \frac{20}{120} \cdot 360° = 60°$

Hip-hop wedge: degrees $= \frac{40}{120} \cdot 360° = 120°$

Example 1.47

Use the data in Example 1.45 and the calculated degrees in Example 1.46 to construct a pie chart.

Solution

Follow the steps listed below:

Step 1: Draw a large circle.

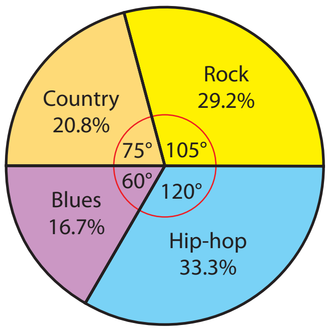
Step 2: Draw a radius (it is usually drawn horizontally).

Step 3: Measure out 105° with a protractor (usually counterclockwise) and make a mark on the circle.

Step 4: Connect the centre of the circle to that point.

Step 5: Using that radius, measure 75° and make a mark on the circle.

Step 6: Connect the centre to that mark and continue until all four angles have been drawn.



The number of degrees should add to 360°.

The percentages should add to 100%. (The total might miss a little due to rounding.)

You do not have to label the wedges with the degrees.

It is customary to label the wedges with a description and with the percentages.

It is also common to shade or colour the wedges for a better visual presentation.

Pictograms

A pictograph (or pictogram), as the name suggests, is a graph that uses pictures to describe or show the data that has been collected.

A general type of picture or symbol is used that best represents the data. For example, basketballs might be used for the number of points scored in a game.

Pictographs have the following characteristics:

- Part, a half or a quarter, of the picture is sometimes used to approximate less than a full amount. For example:
  - If one car represented 1000 accidents, a half of a car would represent 500 accidents.
- A legend is necessary to tell the reader how much of the data the picture represents.
- The scale is usually on the left vertical axis or the bottom horizontal axis.
- Pictographs are easy and fun to look at.

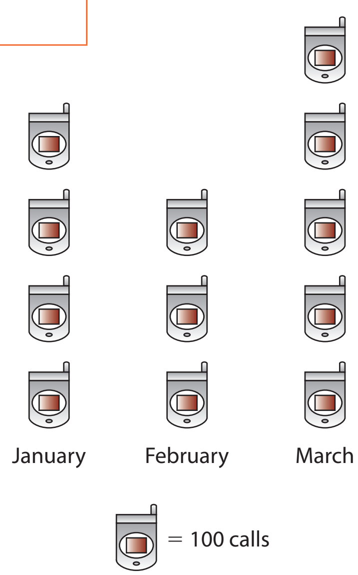
Example 1.48

The following data represents the number of cellphone calls that were made during the first three months of 2008 by a randomly selected household. Construct a pictograph, using a suitable symbol.
1 symbol = 100 calls.

Month	Number of calls
January	400
February	300
March	500

Solution

Number of cellphone calls made during the first three months of 2008. This is an example of a vertical pictograph.



Exercise 1.7

1. Classify each as nominal, ordinal, or interval data.

a) Hair colour
b) Class in high school
c) Weight
d) IQ score
e) Race (Korean, African-American, Hispanic, etc.)
f) Gender
g) Opinion about one's class schedule
h) Music ratings
i) Movie genres
j) Age

2. Construct a bar chart for each set of data below.

a) Last year car Company A sold 600 vehicles, Company B sold 400 vehicles and Company C sold 800 vehicles.

b) Last month Read-a-Lot book store sold the following number of books:

Type	Number sold
Action	350
Non-fiction	200
Romance	400
Science fiction	250

3. Construct a pie chart for each set of data below.

a) It was found that in an elementary school, there were 60 students with blond hair, 130 with brown hair, 80 with black hair and 30 with red hair.

b) A golf store keeps track, on a weekly basis, of the number (in dozens) of inexpensive, moderately priced or expensive golf balls it sells. The data is recorded in the chart below from a randomly selected week.

Price	Number of dozens
Inexpensive	70
Moderately priced	100
Expensive	50

4. Construct a pictograph for each set of data below.

a) The number of buses that a school district used over the last four years is recorded below.
Let 1 bus picture = 20 buses. Construct this as a horizontal graph.

Year	Number of buses
2006	100
2005	80
2004	60
2003	40

b) The All-Sports sporting goods store kept track of how many basketballs it sold during the NBA season. The data from four months is recorded below.
Let one ball picture = 10 basketballs. Construct this as a vertical graph.

Month	Number of basketballs sold
March	40
April	45
May	60
June	75