Ch. 12 Test Review

1) (a) $\overrightarrow{CD} = \overrightarrow{CO} + \overrightarrow{OD} = -\overrightarrow{OC} + \overrightarrow{OD}$ or $\overrightarrow{OD} - \overrightarrow{OC}$

(b) $\overrightarrow{OA} = \frac{1}{2}(\overrightarrow{BA}) = \frac{1}{2}(\overrightarrow{CD}) = \frac{1}{2}(\overrightarrow{OD} - \overrightarrow{OC})$

(c) $\overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OD} = -\overrightarrow{OA} + \overrightarrow{OD} = \overrightarrow{OD} - \overrightarrow{OA} = \overrightarrow{OD} - \frac{1}{2}(\overrightarrow{OD} - \overrightarrow{OC})$

$= \frac{1}{2}\overrightarrow{OD} + \frac{1}{2}\overrightarrow{OC}$

2) $\overrightarrow{OG} = 5i + 5j + 5k$

$\overrightarrow{BD} = -5i + 5k$

$\overrightarrow{EB} = 5i + 5j - 5k$

3) $a(u+v) = 8i + (b-2)j$

$a(3i + 5j + i - 2j) = 8i + (b-2)j$

$a(4i + 3j) = 8i + (b-2)j$

$\underline{4a}i + 3aj = \underline{8}i + (b-2)j$

$4a = 8 \rightarrow 3a = b-2$

$a = 2$ $\quad 3(2) = b-2$

$6 = b-2$

$b = 8$

4) $\begin{pmatrix} -3 \\ m \end{pmatrix} = \begin{pmatrix} 18 \\ -2 \end{pmatrix} + t\begin{pmatrix} -7 \\ 4 \end{pmatrix}$

$-3 = 18 - 7t \Rightarrow t = 3$

$m = -2 + 4t \Rightarrow m = -2 + 4(3)$ $\quad \boxed{m = 10}$

5) $\begin{pmatrix} 3 \\ m \\ n \end{pmatrix} = k\begin{pmatrix} -12 \\ -20 \\ 2 \end{pmatrix} \Rightarrow$

$3 = -12k$ $\quad m = k(-20)$ $\quad n = k(2)$

$k = -1/4$ $\quad m = -1/4(-20)$ $\quad n = (-1/4)(2)$

$\boxed{m = 5}$ $\quad \boxed{n = -1/2}$

6) $\begin{pmatrix} 2-t \\ 3 \\ t \end{pmatrix} \cdot \begin{pmatrix} t \\ 4 \\ t+1 \end{pmatrix} = 0$

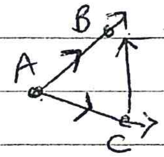
$(2-t)(t) + (3)(4) + (t)(t+1) = 0$

$2t - t^2 + 12 + t^2 + t = 0$

$3t + 12 = 0$

$3t = -12$

$\boxed{t = -4}$

7) 

$$\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB}$$
$$= -\overrightarrow{AC} + \overrightarrow{AB}$$
$$= \overrightarrow{AB} - \overrightarrow{AC}$$

$$\Rightarrow \begin{pmatrix} 2 \\ -7 \\ 4 \end{pmatrix} - \begin{pmatrix} -6 \\ 1 \\ -3 \end{pmatrix} = \boxed{\begin{pmatrix} 8 \\ -8 \\ 7 \end{pmatrix}}$$

8) (a) $u \cdot v = 0$

$(2)(4) + (3)(1) + (-1)(-p) = 0$

$8 + 3 + p = 0$

$11 + p = 0$ $\boxed{p = -11}$

(b) $q|u| = 14$

$q\sqrt{(2)^2 + (3)^2 + (-1)^2} = 14$

$q\sqrt{4 + 9 + 1} = 14$

$q\sqrt{14} = 14$

$\boxed{q = \sqrt{14}}$

9) (a) not same → direction vector not a scalar multiple

(b) same → direction vector is a scalar multiple

(c) not same → direction vector not a scalar multiple

(d) same → direction vector the same and position vector is also on the original line (if $t=1$)

10) (a) $|u| = |v|$

$\sqrt{(-1)^2 + (8)^2 + (-4)^2} = \sqrt{(4)^2 + (4)^2 + (-p)^2}$

$1 + 64 + 16 = 16 + 16 + p^2$

$65 = 16 + p^2$

$49 = p^2$ $\boxed{p = \pm 7}$

(b) $(-1)(4) + (8)(4) + (-4)(-p) = 0$

$-4 + 32 + 4p = 0$

$4p = -28$

$\boxed{p = -7}$

(c) $u - 2v$

$-i + 8j - 4k - 2(4i + 4j - pk)$

$-i + 8j - 4k - 8i - 8j + 2pk$

$-9i + (-4 + 2p)k \parallel i + 2k$

$-9i + (-4 + 2p)k = -9(i + 2k)$

$-9i + (-4 + 2p)k = -9i - 18k$

$-4 + 2p = -18$

$2p = -14$

$\boxed{p = -7}$

11) Subtract the two position vectors (in any order)

$$\begin{pmatrix}-1\\4\end{pmatrix}-\begin{pmatrix}3\\-1\end{pmatrix}=\begin{pmatrix}-4\\5\end{pmatrix}=b \quad \text{or} \quad \text{could have been } \begin{pmatrix}4\\-5\end{pmatrix}$$

Then pick one point - any point $= a$

$r = a + tb$ $\qquad r = \begin{pmatrix}-1\\4\end{pmatrix}+t\begin{pmatrix}-4\\5\end{pmatrix}$ (one possible answer)

12) compare "b" vectors

$$\begin{pmatrix}0\\5\\-2\end{pmatrix} \text{ and } \begin{pmatrix}9\\4\\10\end{pmatrix}$$

parallel — no → not multiples

perp: $(0)(9)+(5)(4)+(-2)(10)=0$

So... $\boxed{\text{perpendicular}}$

13) unit vectors have magnitude of 1

$$\sqrt{k^2+(1/\sqrt{2})^2+(-k)^2}=1$$

$$k^2+1/2+k^2=1$$
$$2k^2=1/2$$
$$4k^2=1$$
$$k^2=1/4 \qquad \boxed{k=\pm 1/2}$$

14) Show 2 of the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, or $\overrightarrow{AC}$ are equal

$\overrightarrow{AB}=\begin{pmatrix}5\\6\end{pmatrix}-\begin{pmatrix}4\\-1\end{pmatrix}=\begin{pmatrix}1\\7\end{pmatrix}$ $\qquad |\overrightarrow{AB}|=\sqrt{1^2+7^2}=\sqrt{50}$

$\overrightarrow{BC}=\begin{pmatrix}1\\3\end{pmatrix}-\begin{pmatrix}5\\6\end{pmatrix}=\begin{pmatrix}-4\\-3\end{pmatrix}$ $\qquad |\overrightarrow{BC}|=\sqrt{(-4)^2+(-3)^2}=\sqrt{25}$

$\overrightarrow{AC}=\begin{pmatrix}1\\3\end{pmatrix}-\begin{pmatrix}4\\-1\end{pmatrix}=\begin{pmatrix}-3\\4\end{pmatrix}$ $\qquad |\overrightarrow{AC}|=\sqrt{(-3)^2+(4)^2}=\sqrt{25}$

Since $\overrightarrow{BC}$ and $\overrightarrow{AC}$ are congruent, $\triangle ABC$ is isosceles.

now in right triangles would have 2 perp. lines and they would be the congruent sides

$$\begin{pmatrix}-4\\-3\end{pmatrix}\begin{pmatrix}-3\\4\end{pmatrix} = (-4)(-3)+(-3)(4) = 12+(-12)=0$$

so... right triangle!

15) collinear → shares a point and parallel

$$\overrightarrow{AB} = \begin{pmatrix}-2\\3\\5\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}-3\\1\\2\end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix}7\\0\\-1\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}6\\-2\\-4\end{pmatrix}$$

share point A

$-2\overrightarrow{AB} = \overrightarrow{AC}$ so parallel

so collinear!

16) (i) $\overrightarrow{PQ} = Q - P$

$$\begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$\boxed{2i + 1j + 2k}$

(ii) $|\overrightarrow{PQ}|$

$$\sqrt{(2)^2 + (1)^2 + (2)^2}$$

$$\sqrt{4+1+4} = \sqrt{9} = \boxed{3}$$

(iii) divide $\overrightarrow{PQ}$ by it's magnitude

$$\frac{1}{3}(2i + 1j + 2k) = \boxed{2/3\, i + 1/3\, j + 2/3\, k}$$

(iv) take unit vector times -5

$$-5\,(2/3\, i + 1/3\, j + 2/3\, k) \Rightarrow \boxed{-10/3\, i - 5/3\, j - 10/3\, k}$$

(b) (i) pt $(3, -1, 3)$ "b" vector $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

$r = a + tb$ $\quad \boxed{r = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}$

(ii) $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = (1)(2) + (2)(1) + (-2)(2)$

$2 + 2 + -4 = 0$ ✓

17) (a) plug in 2 values for s into L_1

$s = 0 \Rightarrow \boxed{\begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix}}$ $\quad s = 1 \Rightarrow \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}}$

(b) $\begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix} + 4\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -8 \\ 8 \\ 4 \end{pmatrix} = \boxed{\begin{pmatrix} -2 \\ 10 \\ 1 \end{pmatrix}}$

(c) $\begin{pmatrix}-2\\10\\1\end{pmatrix} = \begin{pmatrix}0\\-12\\7\end{pmatrix} + t\begin{pmatrix}-1\\11\\-3\end{pmatrix} \Rightarrow$ $-2 = 0 - t \Rightarrow t=2$
$10 = -12 + 11t \Rightarrow t=2$
$1 = 7 - 3t \Rightarrow t=2$

Since all equations give me the same t value P is on L_2.

18) find angle between directional vectors $\begin{pmatrix}4\\3\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$

$\cos\theta = \dfrac{a\cdot b}{|a|\cdot|b|}$

$a\cdot b = (4)(1)+(3)(-1) = 4+-3 = 1$
$|a| = \sqrt{(4)^2+(3)^2} = \sqrt{16+9} = \sqrt{25} = 5$
$|b| = \sqrt{1+1} = \sqrt{2}$

$\cos\theta = \dfrac{1}{5\sqrt{2}} \Rightarrow \cos^{-1}\left(\dfrac{1}{5\sqrt{2}}\right) = 81.9°$

Obtuse angle $= 180 - 81.9° = 98.1°$

19) $2 + 1s = 3 - t \rightarrow s = 1-t$
$5 + 2s = -3+3t$
$3 + 3s = 8 - 4t$

$5+2(1-t) = -3+3t$
$5+2-2t = -3+3t$
$7-2t = -3+3t$
$10 = 5t$
$t=2$ $s = 1-2$
$s=-1$

$\begin{pmatrix}2\\5\\3\end{pmatrix} + (-1)\begin{pmatrix}1\\2\\3\end{pmatrix}$ $\begin{pmatrix}3\\-3\\8\end{pmatrix} + (2)\begin{pmatrix}-1\\3\\-4\end{pmatrix}$

$\begin{pmatrix}2\\5\\3\end{pmatrix} + \begin{pmatrix}-1\\-2\\-3\end{pmatrix} = \boxed{\begin{pmatrix}1\\3\\0\end{pmatrix}}$ $\begin{pmatrix}3\\-3\\8\end{pmatrix} + \begin{pmatrix}-2\\6\\-8\end{pmatrix} = \boxed{\begin{pmatrix}1\\3\\0\end{pmatrix}}$ ← Answer $\boxed{(1,3,0)}$

20) ⓐ $A = (3, 2, 7)$

ⓑ $|\text{velocity}| = \sqrt{(3)^2 + (4)^2 + (10)^2} = \sqrt{9+16+100} = \sqrt{125} = \boxed{11.2 \text{ meters/min}}$

ⓒ $\begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 4 \\ 10 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} + \begin{pmatrix} 6 \\ 8 \\ 20 \end{pmatrix} = \boxed{\begin{pmatrix} 9 \\ 10 \\ 27 \end{pmatrix}}$ ← position

How far? $\sqrt{(9)^2 + (10)^2 + (27)^2} = \sqrt{910} = \boxed{30.2 \text{ meters}}$

ⓓ $t = 0 \Rightarrow$ 13:00 (t = minutes)

$t = -60 \Rightarrow$ 12:00

$$\begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} + (-60)\begin{pmatrix} 3 \\ 4 \\ 10 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} + \begin{pmatrix} -180 \\ -240 \\ -600 \end{pmatrix} = \boxed{\begin{pmatrix} -177 \\ -238 \\ -593 \end{pmatrix}}$$

ⓔ $\begin{pmatrix} 3 \\ 16 \\ 39 \end{pmatrix} - \begin{pmatrix} -5 \\ 10 \\ 23 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 16 \end{pmatrix} \div 2\text{ min} = \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix}$ = velocity vector

position vector = initial position $= \begin{pmatrix} -5 \\ 10 \\ 23 \end{pmatrix}$ $\boxed{r = \begin{pmatrix} -5 \\ 10 \\ 23 \end{pmatrix} + t\begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix}}$

ⓕ plane 2 at $t = 2$

$$\cancel{\begin{pmatrix} -5 \\ 10 \\ 23 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} -5 \\ 10 \\ 23 \end{pmatrix} + \begin{pmatrix} 8 \\ 6 \\ 16 \end{pmatrix} =} \begin{pmatrix} 3 \\ 16 \\ 39 \end{pmatrix}$$

Subtract 2 vectors for $t = 2$

$$\underset{\text{Plane 2 at } t=2}{\begin{pmatrix} 3 \\ 16 \\ 39 \end{pmatrix}} - \underset{\text{Plane 1 at } t=2}{\begin{pmatrix} 9 \\ 10 \\ 27 \end{pmatrix}} = \begin{pmatrix} -6 \\ 6 \\ 12 \end{pmatrix}$$

magnitude

$\sqrt{(-6)^2 + (6)^2 + (12)^2}$

$\sqrt{36 + 36 + 144}$

$\sqrt{216} = \boxed{14.7 \text{ meters}}$

Challenge Question

(a)

$$\begin{pmatrix}-4\\3\end{pmatrix}+t\begin{pmatrix}4\\17\end{pmatrix}=\begin{pmatrix}4\\9\end{pmatrix}+s\begin{pmatrix}-12\\5\end{pmatrix}$$

$-4+4t = 4-12s \rightarrow t=2-3s$

$3+17t = 9+5s \leftarrow$ Substitution

$3+17(2-3s)=9+5s$

$3+34-51s=9+5s$

$28=56s$

$s=1/2$

Plug in s to find both t's

$-4+4t=4-12(1/2) \rightarrow t=0.5$

$3+17t=9+5(1/2) \rightarrow t=0.5$

Since all t's are the same it collides at $t=1/2$ hr. or 12:30 pm

Position: $\begin{pmatrix}-4\\3\end{pmatrix}+(1/2)\begin{pmatrix}4\\17\end{pmatrix}=\begin{pmatrix}-2\\11.5\end{pmatrix}$

(b) at 12:15
$t=1/4$ hr.

$$\begin{pmatrix}-4\\3\end{pmatrix}+(1/4)\begin{pmatrix}4\\17\end{pmatrix}=\begin{pmatrix}-3\\7.25\end{pmatrix}$$

new vector for ship A

$\begin{pmatrix}-3\\7.25\end{pmatrix}+t\begin{pmatrix}16\\17\end{pmatrix}$ where t = time since 12:15

position at 12:30

Ship A: $t=1/4$
use new equation

$$\begin{pmatrix}-3\\7.25\end{pmatrix}+1/4\begin{pmatrix}16\\17\end{pmatrix}=\begin{pmatrix}1\\11.5\end{pmatrix}$$

ship B: $t=1/2$
use B. equation

$$\begin{pmatrix}4\\9\end{pmatrix}+(1/2)\begin{pmatrix}-12\\5\end{pmatrix}=\begin{pmatrix}-2\\11.5\end{pmatrix}$$

distance: $\begin{pmatrix}1\\11.5\end{pmatrix}-\begin{pmatrix}-2\\11.5\end{pmatrix}=\begin{pmatrix}3\\0\end{pmatrix}$ $\sqrt{3^2+0^2}=\boxed{3\text{ km}}$