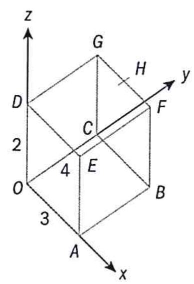
12 Vectors

Skills check



1 **a** $A = (3, 0, 0)$

b $B = (3, 4, 0)$

c $E = (3, 0, 2)$

d $F = (3, 4, 2)$

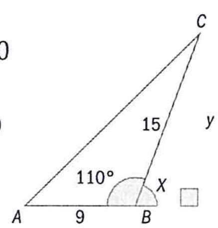
e $H = \left(\frac{3}{2}, 4, 2\right)$

2 $x^2 = 3^2 + 6^2$

$= 9 + 36$

$= 45$

$x = \sqrt{45} \approx 6.71$



3 **a** $X = 180 - 110 = 70$

$\cos X = \frac{z}{15} \Rightarrow z = 15\cos 70$

≈ 5.13

$\sin X = \frac{y}{15} \Rightarrow y = 15\sin 70$

≈ 14.1

$(AC)^2 = y^2 + (9+z)^2$

$= (14.1)^2 + (9+5.13)^2$

$AC = \sqrt{432.5}$

$= 20.8$

$= 21\,\text{cm}$ (to the nearest centimetre)

b Using the Cosine Rule

$(AC)^2 = (AB)^2 + (BC)^2 - 2(AB)(BC)\cos(A\hat{B}C)$

$(9.7)^2 = (8.6)^2 + (3.1)^2 - 2(8.6)(3.1)\cos(A\hat{B}C)$

$A\hat{B}C = \cos^{-1}\left[\frac{(8.6)^2 + (3.1)^2 - (9.7)^2}{2(8.6)(3.1)}\right]$

$\approx 101.4°$

Exercise 12A

1 **a** $\mathbf{x} = -2\mathbf{i} + 3\mathbf{j}$

b $\mathbf{y} = 7\mathbf{j}$

c $\mathbf{z} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

2 **a** $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

b $\overrightarrow{CD} = \begin{pmatrix} -1 \\ 6 \\ -1 \end{pmatrix}$

c $\overrightarrow{EF} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

3 **a** $= -3\mathbf{i} - 5\mathbf{j} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$

b $= -2\mathbf{i} + 4\mathbf{j} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

c $= 3\mathbf{i} + 8\mathbf{j} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$

d $= 6\mathbf{j} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$

e $= -3\mathbf{i} - 6\mathbf{j} = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$

4 **a** $\left|\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

b $\left|\begin{pmatrix} 1 \\ -3 \end{pmatrix}\right| = \sqrt{1^2 + (-3)^2} = \sqrt{10} \approx 3.16$

c $|2\mathbf{i} + 5\mathbf{j}| = \sqrt{2^2 + 5^2} = \sqrt{29} \approx 5.39$

d $\left|\begin{pmatrix} 2.8 \\ 4.5 \end{pmatrix}\right| = \sqrt{(2.8)^2 + (4.5)^2} = 5.3$

e $|2\mathbf{i} - 5\mathbf{j}| = \sqrt{2^2 + (-5)^2} = \sqrt{29} \approx 5.39$

5 **a** $\left|\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}\right| = \sqrt{3^2 + 2^2 + 5^2} = \sqrt{38} \approx 6.16$

b $\left|\begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}\right| = \sqrt{4^2 + (-1)^2 + (-3)^2} = \sqrt{26} \approx 5.10$

c $|2\mathbf{i} + 2\mathbf{j} + \mathbf{k}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$

d $\left|\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix}\right| = \sqrt{(-3)^2 + 2^2 + 6^2} = \sqrt{49} = 7$

e $|\mathbf{j} - \mathbf{k}| = \sqrt{1^2 + (-1)^2} = \sqrt{2} \approx 1.41$

Exercise 12B

1 **a** $= \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\mathbf{c} = \begin{pmatrix} -6 \\ 3 \end{pmatrix} = -3\begin{pmatrix} 2 \\ -1 \end{pmatrix} = -3\mathbf{b}$

$\mathbf{d} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \frac{1}{2}\mathbf{a}$

$\mathbf{e} = \begin{pmatrix} -10 \\ 5 \end{pmatrix} = -5\begin{pmatrix} 2 \\ -1 \end{pmatrix} = -5\mathbf{b}$

$\mathbf{f} = \begin{pmatrix} -4 \\ -8 \end{pmatrix} = -2\begin{pmatrix} 2 \\ 4 \end{pmatrix} = -2\mathbf{a}$

We must have s & t so that

$-5 = 2s + 2t \quad (1)$

$-8 = 4s - t \quad (2)$

$2 \times (2): -16 = 8s - 2t \quad (3)$

$(1) + (3): -21 = 10s$

$s = \frac{-21}{10}$

from (2): $-8 = \frac{-84}{10} - t$

$t = 8 - \frac{84}{10}$

$= \frac{-2}{5}$

so $\mathbf{f} = -2\begin{pmatrix} 2 \\ 4 \end{pmatrix} -2\mathbf{a}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

2 a $\mathbf{a} = \begin{pmatrix} 0.1 \\ 0.7 \end{pmatrix}$

$= \frac{1}{10}\begin{pmatrix} 1 \\ 7 \end{pmatrix}$

$= \frac{1}{10}(\mathbf{i} + 7\mathbf{j})$

a is parallel to $\mathbf{i} + 7\mathbf{j}$ with $\frac{1}{10}$ the magnitude.

$\mathbf{b} = \begin{pmatrix} -1 \\ -7 \end{pmatrix}$

$= -1(\mathbf{i} + 7\mathbf{j})$

b is parallel to $(\mathbf{i} + 7\mathbf{j})$ with opposite direction.

$\mathbf{c} = \begin{pmatrix} -0.05 \\ -0.03 \end{pmatrix}$ is not parallel to $(\mathbf{i} + 7\mathbf{j})$

$\mathbf{d} = \begin{pmatrix} -10 \\ 70 \end{pmatrix}$ is not parallel to $(\mathbf{i} + 7\mathbf{j})$

$\mathbf{e} = 60\mathbf{i} + 420\mathbf{j}$

$= 60(\mathbf{i} + 7\mathbf{j})$

e is parallel to $(\mathbf{i} + 7\mathbf{j})$ with 60 times the magnitude

$\mathbf{f} = (6\mathbf{i} - 42\mathbf{j})$ is not parallel to $(\mathbf{i} + 7\mathbf{j})$

$\mathbf{g} = (-\mathbf{i} + 7\mathbf{j})$ is not parallel to $(\mathbf{i} + 7\mathbf{j})$

3 a For parallel vectors, $\mathbf{r} = k\,\mathbf{s}$ for some k

$(4\mathbf{i} + t\mathbf{j}) = k(14\mathbf{i} - 12\mathbf{j})$

$4 = 14k$

$k = \frac{4}{14}$

$= \frac{2}{7}$

$t = -12k$

$= -12 \times \frac{2}{7}$

$= \frac{-24}{7}$

b For parallel vectors, $\mathbf{a} = k\,\mathbf{b}$ for some k

$\begin{pmatrix} t \\ -8 \end{pmatrix} = k\begin{pmatrix} 7 \\ -10 \end{pmatrix}$

$-8 = -10k$

$k = \frac{-8}{-10}$

$= \frac{4}{5}.$

so $t = 7k$

$= 7\left(\frac{4}{5}\right)$

$= \frac{28}{5}$

4 For parallel vectors, $\mathbf{v} = k\,\mathbf{w}$ for some k

$t\mathbf{i} - 5\mathbf{j} + 8\mathbf{k} = k(5\mathbf{i} + \mathbf{j} + s\,\mathbf{k})$

$-5 = k$

so $t = 5k$

$t = 5(-5)$

$= -25$

$8 = sk = (-5)s$

$s = \frac{-8}{5}$

5 a $\overrightarrow{OG} = \mathbf{j} + \mathbf{k}$

b $\overrightarrow{BD} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$

c $\overrightarrow{AD} = -\mathbf{i} + \mathbf{k}$

d $\overrightarrow{OM} = \frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{k}$

6 a $\overrightarrow{OG} = 4\mathbf{j} + 3\mathbf{k}$

b $\overrightarrow{BD} = -5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

c $\overrightarrow{AD} = -5\mathbf{i} + 3\mathbf{k}$

d $\overrightarrow{OM} = \frac{5}{2}\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$

Exercise 12C

1 $\overrightarrow{OP} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} \quad \overrightarrow{OQ} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$$

$$\overrightarrow{QP} = -\overrightarrow{PQ} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

2 $A = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad C = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

a $\overrightarrow{AB} = B - A = \begin{pmatrix} 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$

b $\overrightarrow{BA} = -\overrightarrow{AB} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

c $\overrightarrow{AC} = C - A = \begin{pmatrix} -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ 3 \end{pmatrix}$

d $\overrightarrow{CB} = B - C = \begin{pmatrix} 1 \\ -3 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$

3 **a** $\overrightarrow{OP} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

b vector is $-\begin{pmatrix} 1 \\ -5 \\ 6 \end{pmatrix} = -\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$

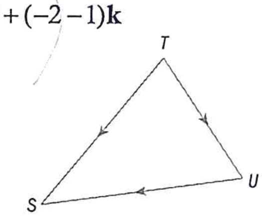
c vector is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -6 \end{pmatrix} = -\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$

d vector is $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 6 \end{pmatrix} = \mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$

4 $\overrightarrow{LM} = \overrightarrow{LN} + \overrightarrow{NM} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ -3 \end{pmatrix}$

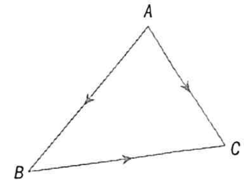
5 From the diagram, we see

$$\overrightarrow{US} = -\overrightarrow{TU} + \overrightarrow{TS}$$
$$= -(\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) + (3\mathbf{i} + 4\mathbf{j} - \mathbf{k})$$
$$= (-1+3)\mathbf{i} + (4+4)\mathbf{j} + (-2-1)\mathbf{k}$$
$$= 2\mathbf{i} + 8\mathbf{j} - 3\mathbf{k}$$



6 From the diagram,

$\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = 0$



$$\begin{pmatrix} 1 \\ y \\ -2 \end{pmatrix} + \begin{pmatrix} 2x \\ -3 \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ x+y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$1 + 2x - 1 = 0 \Rightarrow 0 + 2x = 0$ (1)

$y - 3 - 4 = 0 \Rightarrow y - 7 = 0$ (2)

$-2 + z - (x + y) \Rightarrow -x - y + z - 2 = 0$ (3)

(1) $\Rightarrow x = 0$

(2) $\Rightarrow y = 7$

(3) $\Rightarrow -2 + z - 7 = 0$

$z = 9$

Exercise 12D

1 $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) - (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$

$= (-2-1)\mathbf{i} + (3-(-2))\mathbf{j} + (-1-3)\mathbf{k}$

$= -3\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (4\mathbf{i} - 7\mathbf{j} + 7\mathbf{k}) - (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$

$= (4-1)\mathbf{i} + (-7-(-2))\mathbf{j} + (7-3)\mathbf{k}$

$= 3\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$

we see $\overrightarrow{AB} = -\overrightarrow{AC}$, so $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel. Since they contain a common point A, they must lie on the same line.

2 **a** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 8 \end{pmatrix}$

b $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 8 \\ -1 \\ 13 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ 16 \end{pmatrix}$

we see $\overrightarrow{AC} = 2\overrightarrow{AB}$, so $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are parallel. Since they contain a common point A, then A, B, & C are collinear.

3 $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$

$\overrightarrow{P_1P_3} = \overrightarrow{OP_3} - \overrightarrow{OP_1} = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \\ 0 \end{pmatrix}$

we see $\overrightarrow{P_1P_3} = 2\overrightarrow{P_1P_2}$. Since they contain a common point, they are collinear.

Since P_4 collinear with P_1, P_2, P_3, we have $\overrightarrow{P_1P_4} = k\overrightarrow{P_1P_2}$ for some $k \in R$

$\overrightarrow{P_1P_4} = \begin{pmatrix} 2 \\ s \\ t \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ s-2 \\ t-4 \end{pmatrix}$ for some s & t

Now $\begin{pmatrix}1\\s-2\\t-4\end{pmatrix}=k\begin{pmatrix}-3\\-1\\0\end{pmatrix}$

$1=-3k\Rightarrow k=\frac{-1}{3}$

$s-2=-k\Rightarrow s=2-k=2+\frac{1}{3}=\frac{7}{3}$

$t-4=0\Rightarrow t=4$

$\therefore P_4=\left(2,\frac{7}{3},4\right)$

4 $\overrightarrow{OA}=3\mathbf{i}+4\mathbf{j},\ \overrightarrow{OB}=x\mathbf{i},\ \overrightarrow{OC}=\mathbf{i}-2\mathbf{j}$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(x-3)\mathbf{i}-4\mathbf{j}$

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(1-3)\mathbf{i}+(-2-4)\mathbf{j}$

$=-2\mathbf{i}-6\mathbf{j}$

If A, B, C are collinear, $\overrightarrow{AB}=k\overrightarrow{AC}$ for some $k\in R$

$\therefore (x-3)\mathbf{i}-4\mathbf{j}=k(-2\mathbf{i}-6\mathbf{j})$

$\mathbf{j}$ components $\Rightarrow -4=-6k\Rightarrow k=\frac{2}{3}$

so $x-3=-2k=\frac{-4}{3}$

$x=\frac{9}{3}-\frac{4}{3}=\frac{5}{3}$

so $\overrightarrow{AB}=\frac{-4}{3}\mathbf{i}-4\mathbf{j}$

$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}$

$=(\mathbf{i}-2\mathbf{j})-\frac{5}{3}\mathbf{i}=\frac{-2}{3}\mathbf{i}-2\mathbf{j}$

$\overrightarrow{AB}:\overrightarrow{BC}=\begin{pmatrix}\frac{-4}{3}\\-4\end{pmatrix}:\begin{pmatrix}\frac{-2}{3}\\-2\end{pmatrix}$

$=2:1$

Exercise 12E

1 $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}4\\5\\-1\end{pmatrix}-\begin{pmatrix}-1\\5\\1\end{pmatrix}=\begin{pmatrix}5\\0\\-2\end{pmatrix}$

Distance $AB=\sqrt{5^2+(-2)^2}$

$=\sqrt{29}$

≈ 5.39

2 $\overrightarrow{AB}=\begin{pmatrix}6\\0\\6\end{pmatrix}-\begin{pmatrix}-5\\2\\4\end{pmatrix}=\begin{pmatrix}11\\-2\\2\end{pmatrix}$

Distance $AB=\sqrt{11^2+2^2+2^2}=\sqrt{129}$

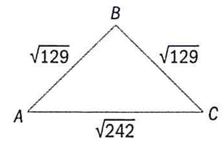
$\overrightarrow{AC}=\begin{pmatrix}8\\10\\1\end{pmatrix}-\begin{pmatrix}-5\\2\\4\end{pmatrix}=\begin{pmatrix}13\\8\\-3\end{pmatrix}$

Distance $AC=\sqrt{13^2+8^2+(-3)^2}=\sqrt{242}$

$\overrightarrow{BC}=\begin{pmatrix}8\\10\\1\end{pmatrix}-\begin{pmatrix}6\\0\\6\end{pmatrix}=\begin{pmatrix}2\\10\\-5\end{pmatrix}$

Distance $BC=\sqrt{2^2+10^2+5^2}=\sqrt{129}$

Distance AB = Distance BC, so ABC is isosceles.



$\cos\left(C\hat{A}B\right)=\frac{129+242-129}{2\sqrt{129}\sqrt{242}}$

$C\hat{A}B=46.8°$

3 $|\mathbf{a}|=7$, so $\sqrt{2^2+(-3)^2+t^2}=7$

$4+9+t^2=49$

$t^2=36$

$t=\pm 6$

4 $\mathbf{a}=x\mathbf{i}+6\mathbf{j}-2\mathbf{k}$

$|\mathbf{a}|=\sqrt{x^2+6^2+(-2)^2}=3x$

$x^2+36+4=9x^2$

$8x^2=40$

$x^2=\pm\sqrt{5}$

5 $|\mathbf{u}|=|\mathbf{v}|$, so

$a^2+(-a)^2+(2a)^2=2^2+(-4)^2+(-2)^2$

$a^2+a^2+4a^2=4+16+4$

$6a^2=24$

$a^2=4$

$a=\pm 2$

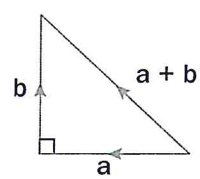
6 **a** $\mathbf{b}=2\mathbf{a}$

Then $|\mathbf{a}+\mathbf{b}|=|3\mathbf{a}|=3|\mathbf{a}|=15$

b $\mathbf{b}=-3\mathbf{a}$

Then $|\mathbf{a}+\mathbf{b}|=|-2\mathbf{a}|=2|\mathbf{a}|=10$

c



Using Pythagoras

$|\mathbf{a}|^2+|\mathbf{b}|^2=|\mathbf{a}+\mathbf{b}|^2$

Hence $|\mathbf{a}+\mathbf{b}|=\sqrt{5^2+12^2}=13$

Exercise 12F

1 $\left|\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}\right|=\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}=\sqrt{\frac{9}{25}+\frac{16}{25}}=\sqrt{1}=1$

2 $\left|\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}\right|=\sqrt{\frac{1}{3^2}+\frac{2^2}{3^2}+\frac{2^2}{3^2}}$

$=\sqrt{\frac{1}{9}+\frac{4}{9}+\frac{4}{9}}$

$=\sqrt{1}=1$

3 $|4\mathbf{i}-3\mathbf{j}|=\sqrt{4^2+(-3)^2}=\sqrt{25}=5$

So unit vector is $\frac{1}{5}(4\mathbf{i}-3\mathbf{j})=\frac{4}{5}\mathbf{i}-\frac{3}{5}\mathbf{j}$

4 $\left|\begin{pmatrix}-1\\-5\\4\end{pmatrix}\right| = \sqrt{(-1)^2+(-5)^2+4^2} = \sqrt{42}$

So unit vector is $\frac{1}{\sqrt{42}}\begin{pmatrix}-1\\-5\\4\end{pmatrix}$

5 $\overline{P_1P_2} = \begin{pmatrix}3\\2\\0\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}2\\2\\-1\end{pmatrix}$

$\left|\overline{P_1P_2}\right| = \sqrt{2^2+2^2+(-1)^2} = \sqrt{9} = 3$

So unit vector is $\frac{1}{3}\begin{pmatrix}2\\2\\-1\end{pmatrix}$

6 $|a\mathbf{i}+2a\mathbf{j}| = \sqrt{a^2+(2a)^2} = \sqrt{5a^2}$

$= \sqrt{5}a$

Now $\sqrt{5}a = 1$, so $a = \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

7 $|2\mathbf{i}-\mathbf{j}| = \sqrt{2^2+(-1)^2}$

$= \sqrt{5}$

So unit vector is $\frac{1}{\sqrt{5}}(2\mathbf{i}-\mathbf{j})$

Vector of magnitude 5 is $\frac{5}{\sqrt{5}}(2\mathbf{i}-\mathbf{j}) = \sqrt{5}(2\mathbf{i}-\mathbf{j})$

8 $\left|\begin{pmatrix}-1\\-3\\2\end{pmatrix}\right| = \sqrt{(-1)^2(-3)^2+2^2} = \sqrt{14}$

unit vector is $\frac{1}{\sqrt{14}}\begin{pmatrix}-1\\-3\\2\end{pmatrix}$

and vector magnitude 7 is $\frac{7}{\sqrt{14}}\begin{pmatrix}-1\\-3\\2\end{pmatrix} = \frac{\sqrt{14}}{2}\begin{pmatrix}-1\\-3\\2\end{pmatrix}$

9 **a** $\left|\begin{pmatrix}2\cos\theta\\2\sin\theta\end{pmatrix}\right| = \sqrt{2^2\cos^2\theta+2^2\sin^2\theta}$

$= \sqrt{4(\cos^2\theta+\sin^2\theta)}$

$= 2\sqrt{(1)} = 2$

So unit vector is $\frac{1}{2}\begin{pmatrix}2\cos\theta\\2\sin\theta\end{pmatrix}$ or $\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}$

b $\left|\begin{pmatrix}1\\\tan\alpha\end{pmatrix}\right| = \sqrt{1^2+\tan^2\alpha}$

$= \sqrt{\sec^2\alpha}$

$= \sec\alpha = \frac{1}{\cos\alpha}$

So unit vector is $\frac{1}{\sec\alpha}\begin{pmatrix}1\\\tan\alpha\end{pmatrix} = \begin{pmatrix}\cos\alpha\\\sin\alpha\end{pmatrix}$

Exercise 12G

1 **a** $\mathbf{a}+\mathbf{b} = (2\mathbf{i}-\mathbf{j})+(3\mathbf{i}+2\mathbf{j})$

$= (2+3)\mathbf{i}+(-1+2)\mathbf{j}$

$= 5\mathbf{i}+\mathbf{j}$

b $\mathbf{b}+\mathbf{c} = (3\mathbf{i}+2\mathbf{j})+(-\mathbf{i}+\mathbf{j})$

$= (3-1)\mathbf{i}+(2+1)\mathbf{j}$

$= 2\mathbf{i}+3\mathbf{j}$

c $\mathbf{c}+\mathbf{d} = (-\mathbf{i}+\mathbf{j})+(3\mathbf{i}+3\mathbf{j})$

$= (-1+3)\mathbf{i}+(1+3)\mathbf{j}$

$= 2\mathbf{i}+4\mathbf{j}$

d $\mathbf{a}+\mathbf{b}+\mathbf{d} = (2\mathbf{i}-\mathbf{j})+(3\mathbf{i}+2\mathbf{j})+(3\mathbf{i}+3\mathbf{j})$

$= (2+3+3)\mathbf{i}+(-1+2+3)\mathbf{j}$

$= 8\mathbf{i}+4\mathbf{j}$

e $\mathbf{a}-\mathbf{b} = (2\mathbf{i}-\mathbf{j})-(3\mathbf{i}+2\mathbf{j})$

$= (2-3)\mathbf{i}+(-1-2)\mathbf{j}$

$= -\mathbf{i}-3\mathbf{j}$

f $\mathbf{d}-\mathbf{b}+\mathbf{a} = (3\mathbf{i}+3\mathbf{j})-(3\mathbf{i}+2\mathbf{j})+(2\mathbf{i}-\mathbf{j})$

$= (3-3+2)\mathbf{i}+(3-2-1)\mathbf{j}$

$= 2\mathbf{i}+0\mathbf{j} = 2\mathbf{i}$

2 **a** $\mathbf{a}+\mathbf{b} = \begin{pmatrix}2\\-3\end{pmatrix}+\begin{pmatrix}-4\\5\end{pmatrix} = \begin{pmatrix}2 & -4\\-3 & +5\end{pmatrix} = \begin{pmatrix}-2\\2\end{pmatrix}$

b $\mathbf{b}-\mathbf{c} = \begin{pmatrix}-4\\5\end{pmatrix}-\begin{pmatrix}-5\\-3\end{pmatrix} = \begin{pmatrix}-4 & -(-5)\\5 & -(-3)\end{pmatrix} = \begin{pmatrix}1\\8\end{pmatrix}$

c $\frac{1}{2}(\mathbf{a}+\mathbf{c}) = \frac{1}{2}\left[\begin{pmatrix}2\\-3\end{pmatrix}+\begin{pmatrix}-5\\-3\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix}2 & -5\\-3 & +(-3)\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-3\\-6\end{pmatrix}$

d $\mathbf{a}+3\mathbf{b}-\mathbf{c} = \begin{pmatrix}2\\-3\end{pmatrix}+3\begin{pmatrix}-4\\5\end{pmatrix}-\begin{pmatrix}-5\\-3\end{pmatrix}$

$= \begin{pmatrix}2+3(-4)-(-5)\\-3+3(5)-(-3)\end{pmatrix}$

$= \begin{pmatrix}-5\\15\end{pmatrix}$

e $3\mathbf{c} - 2\mathbf{b} + 5\mathbf{a} = 3\begin{pmatrix}-5\\-3\end{pmatrix} - 2\begin{pmatrix}-4\\5\end{pmatrix} + 5\begin{pmatrix}2\\-3\end{pmatrix}$

$= \begin{pmatrix}3(-5) - 2(-4) + 5(2)\\3(-3) - 2(5) + 5(-3)\end{pmatrix}$

$= \begin{pmatrix}-15 + 8 + 10\\-9 - 10 - 15\end{pmatrix}$

$= \begin{pmatrix}3\\-34\end{pmatrix}$

3 **a** $\mathbf{a} + \mathbf{b} = (3\mathbf{i} - \mathbf{j} - 2\mathbf{k}) + (5\mathbf{i} - \mathbf{k})$

$= (3 + 5)\mathbf{i} + (-1)\mathbf{j} + (-2 - 1)\mathbf{k}$

$= 8\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

b $\mathbf{b} - 2\mathbf{a} = (5\mathbf{i} - \mathbf{k}) - 2(3\mathbf{i} - \mathbf{j} - 2\mathbf{k})$

$= (5 - 6)\mathbf{i} - 2(-1)\mathbf{j} + (-1 - 2(-2))\mathbf{k}$

$= -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

c $2\mathbf{a} - \mathbf{b} = 2(3\mathbf{i} - \mathbf{j} - 2\mathbf{k}) - (5\mathbf{i} - \mathbf{k})$

$= (6 - 5)\mathbf{i} + (-2)\mathbf{j} + (-4 + 1)\mathbf{k}$

$= \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$

d $4(\mathbf{a} - \mathbf{b}) + 2(\mathbf{b} + \mathbf{a}) = 4((3 - 5)\mathbf{i} - \mathbf{j} + (-2 + 1)\mathbf{k})$

$+ 2(8\mathbf{i} - \mathbf{j} - 3\mathbf{k})$ from Q3a

$= -8\mathbf{i} - 4\mathbf{j} - 4\mathbf{k} + 16\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$

$= (-8 + 16)\mathbf{i} - (4 + 2)\mathbf{j} + (-4 - 6)\mathbf{k}$

$= 8\mathbf{i} - 6\mathbf{j} - 10\mathbf{k}$

4 **a** $2\mathbf{x} - 3\mathbf{p} = \mathbf{q}$

$2\mathbf{x} - 3\begin{pmatrix}3\\-5\end{pmatrix} = \begin{pmatrix}-1\\4\end{pmatrix}$

$2\mathbf{x} = \begin{pmatrix}-1 + 9\\4 - 15\end{pmatrix} = \begin{pmatrix}8\\-11\end{pmatrix}$

$\mathbf{x} = \begin{pmatrix}4\\-5.5\end{pmatrix}$

b $4\begin{pmatrix}3\\-5\end{pmatrix} - 3(\mathbf{y}) = 7\begin{pmatrix}-1\\4\end{pmatrix}$

$\begin{pmatrix}12\\-20\end{pmatrix} - 3\mathbf{y} = \begin{pmatrix}-7\\28\end{pmatrix}$

$\begin{pmatrix}12\\-20\end{pmatrix} - \begin{pmatrix}-7\\28\end{pmatrix} = 3\mathbf{y}$

$\begin{pmatrix}19\\-48\end{pmatrix} = 3\mathbf{y}$

So $\mathbf{y} = \frac{1}{3}(19\mathbf{i} - 48\mathbf{j})$

c $2\mathbf{p} + \mathbf{z} = 0$

$2\begin{pmatrix}3\\-5\end{pmatrix} + \begin{pmatrix}z_1\\z_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$6 + z_1 = 0 \Rightarrow z_1 = -6$

$-10 + z_2 = 0 \Rightarrow z_2 = 10$

$\mathbf{z} = -6\mathbf{i} + 10\mathbf{j}$

5 $\mathbf{a} = \mathbf{b} \Rightarrow \begin{pmatrix}x\\x + y\end{pmatrix} = \begin{pmatrix}6 - y\\-2x - 3\end{pmatrix}$

$x = 6 - y$ (1)

$x + y = -2x - 3$

$y = -3x - 3$ (2)

Sub (1) into (2)

$y = -3(6 - y) - 3$

$y = -18 + 3y - 3$

$-2y = -21$

$y = \frac{21}{2}$

$x = 6 - y = 6 - \left(\frac{21}{2}\right) = \frac{-9}{2}$

6 $3\mathbf{a} = 2\mathbf{b} \Rightarrow 3\begin{pmatrix}3\\t\\u\end{pmatrix} = 2\begin{pmatrix}t - s\\3s\\t + s\end{pmatrix}$

(1) $9 = 2(t - s)$

(2) $3t = 6s$

(3) $3u = 2(t + s)$

(2) $\Rightarrow t = 2s$

(1) $\Rightarrow 9 = 2(2s - s)$

$= 2s$

$s = \frac{9}{2}$

(2) $\Rightarrow t = 9$

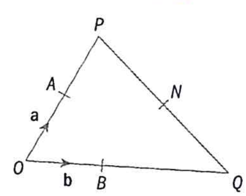
(3) $\Rightarrow 3u = 2\left(9 + \frac{9}{2}\right)$

$u = \frac{27}{3} = 9$

$t = 9, s = \frac{9}{2}, u = 9$

Exercise 12H

1



a $\overline{AP} = \overline{OA} = \mathbf{a}$

b $\overline{AB} = -\overline{OA} + \overline{OB}$

$= -\mathbf{a} + \mathbf{b}$

$= \mathbf{b} - \mathbf{a}$

c $\overline{PQ} = -\overline{AP} - \overline{OA} + \overline{OB} + \overline{BQ}$
$= -\mathbf{a} - \mathbf{a} + \mathbf{b} + 3\mathbf{b}$
$= 4\mathbf{b} - 2\mathbf{a}$

d $\overline{PN} = \frac{1}{2}\overline{PQ} = \frac{1}{2}(4\mathbf{b} - 2\mathbf{a})$
$= 2\mathbf{b} - \mathbf{a}$

e $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AP} + \overrightarrow{PN}$
$= \mathbf{a} + \mathbf{a} + (2\mathbf{b} - \mathbf{a})$
$= \mathbf{a} + 2\mathbf{b}$

f $\overline{AN} = \overline{AP} + \overline{PN}$
$= \mathbf{a} + (2\mathbf{b} - \mathbf{a})$
$= 2\mathbf{b}$

2 $\mathbf{a} = \overline{OA}, \mathbf{b} = \overline{OB}, \overline{AC} : \overline{CB} = 3:1$

a $\overline{AB} = -\overline{OA} + \overline{OB}$
$= -\mathbf{a} + \mathbf{b}$
$= \mathbf{b} - \mathbf{a}$

b $\overline{AC} = \frac{3}{4}\overline{AB}$
$= \frac{3}{4}(\mathbf{b} - \mathbf{a})$

c $\overline{CB} = \frac{1}{4}\overline{AB}$
$= \frac{1}{4}(\mathbf{b} - \mathbf{a})$

d $\overline{OC} = \overline{OA} + \overline{AC}$
$= \mathbf{a} + \frac{3}{4}(\mathbf{b} - \mathbf{a})$
$= \mathbf{a}\left(1 - \frac{3}{4}\right) + \mathbf{b}\frac{3}{4}$
$= \frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}$

3 $\overline{OA} = \mathbf{a}, \overline{OC} = \mathbf{c}, \overline{CB} = 3\mathbf{a}$

a $\overline{OB} = \overline{OC} + \overline{CB}$
$= \mathbf{c} + 3\mathbf{a}$

b $\overline{AB} = -\overline{OA} + \overline{OC} + \overline{CB}$
$= -\mathbf{a} + \mathbf{c} + 3\mathbf{a}$
$= \mathbf{c} + 2\mathbf{a}$

c $\overline{OD} = \overline{OA} + \frac{1}{2}\overline{AB}$
$= \mathbf{a} + \frac{1}{2}(\mathbf{c} + 2\mathbf{a})$
$= 2\mathbf{a} + \frac{1}{2}\mathbf{c}$

d $\overline{CD} = \overline{CB} - \frac{1}{2}\overline{AB}$
$= 3\mathbf{a} - \frac{1}{2}(\mathbf{c} + 2\mathbf{a})$
$= 2\mathbf{a} - \frac{1}{2}\mathbf{c}$

4 a $\overline{FA} = \mathbf{a}, \overline{FB} = \mathbf{b}$

i $\overline{AB} = -\overline{FA} + \overline{FB}$
$= -\mathbf{a} + \mathbf{b}$

ii $\overline{FO} = \overline{FB} + \overline{BO}$
By symmetry, $\overline{OB} = \overline{FA} = \mathbf{a}$
so $\overline{FO} = \overline{FB} - \overline{OB}$
$= \mathbf{b} - \mathbf{a}$

iii $\overline{FC} = \overline{FO} + \overline{OE} + \overline{EC}$
By symmetry, $\overline{OE} = \overline{BO} = -\mathbf{a}$
and $\overline{EC} = \overline{FB} = \mathbf{b}$
so $\overline{FC} = (\mathbf{b} - \mathbf{a}) - \mathbf{a} + \mathbf{b}$
$= 2(\mathbf{b} - \mathbf{a})$

iv $\overline{BC} = \overline{BE} + \overline{EC}$
$= \overline{BO} + \overline{OC} + \overline{EC}$
$= -\overline{OB} + \overline{OC} + \overline{EC}$
$= -\mathbf{a} - \mathbf{a} + \mathbf{b}$
$= \mathbf{b} - 2\mathbf{a}$

v $\overline{FD} = \overline{FC} + \overline{CD}$
By symmetry, $\overline{CD} = -\overline{FA} = -\mathbf{a}$
so $\overline{FD} = 2(\mathbf{b} - \mathbf{a}) - \mathbf{a} = 2\mathbf{b} - 3\mathbf{a}$

b AB is parallel to and half the length of FC

c $\overline{FD} = 2\mathbf{b} - 3\mathbf{a}$
$\overline{AC} = \overline{AF} + \overline{FC}$ (see **iii**)
$= -\mathbf{a} + 2(\mathbf{b} - \mathbf{a})$
$= -3\mathbf{a} + 2\mathbf{b}$
$\overline{FD} = \overline{AC}$ $\therefore$ FD and AC are parallel

5 $\overline{OA} = \mathbf{a}, \overline{OB} = \mathbf{b}$

a $\overline{AB} = -\overline{OA} + \overline{OB}$
$= -\mathbf{a} + \mathbf{b}$
$= \mathbf{b} - \mathbf{a}$
since $\overline{AP} = \frac{2}{3}\overline{AB}$
$\overline{AP} = \frac{2}{3}(\mathbf{b} - \mathbf{a})$

b M is mid point of OA, so
$\overline{MA} = \frac{1}{2}\overline{OA} = \frac{1}{2}\mathbf{a}$
$\overline{MP} = \overline{MA} + \overline{AP}$
$= \frac{1}{2}\mathbf{a} + \frac{2}{3}(\mathbf{b} - \mathbf{a})$
$= \left(\frac{1}{2} - \frac{2}{3}\right)\mathbf{a} + \frac{2}{3}\mathbf{b}$
$= \frac{2}{3}\mathbf{b} - \frac{1}{6}\mathbf{a}$

c $\overline{MX} = \overline{MP} + \overline{PB} + \overline{BX}$
$\overline{PB} = \frac{1}{3}\overline{AB} = \frac{1}{3}(\mathbf{b} - \mathbf{a})$
$\overline{BX} = \overline{OB} = \mathbf{b}$
so $\overline{MX} = \left(\frac{2}{3}\mathbf{b} - \frac{1}{6}\mathbf{a}\right) + \frac{1}{3}(\mathbf{b} - \mathbf{a}) + \mathbf{b}$
$= 2\mathbf{b} - \frac{1}{2}\mathbf{a}$

d $\overline{MP} = \frac{2}{3}\mathbf{b} - \frac{1}{6}\mathbf{a}$

$\overline{MX} = 2\mathbf{b} - \frac{1}{2}\mathbf{a}$

$\overline{MX} = 3\overline{MP}$ $\therefore$ MX is parallel to MP

Since MX and MP share the common point M, MPX is a straight line

Exercise 12I

1 a $\mathbf{a}\cdot\mathbf{b} = (2\mathbf{i}+4\mathbf{j})\cdot(\mathbf{i}-5\mathbf{j})$
$= (2\times1)+(4\times-5)$
$= 2-20$
$= -18$

b $\mathbf{b}\cdot\mathbf{c} = (\mathbf{i}-5\mathbf{j})\cdot(-5\mathbf{i}-2\mathbf{j})$
$= (1\times-5)+(-5\times-2)$
$= -5+10$
$= 5$

c $\mathbf{a}\cdot\mathbf{a} = (2\mathbf{i}+4\mathbf{j})\cdot(2\mathbf{i}+4\mathbf{j})$
$= (2\times2)+(4\times4)$
$= 4+16$
$= 20$

d $\mathbf{c}\cdot(\mathbf{a}+\mathbf{b}) = (-5\mathbf{i}-2\mathbf{j})\cdot[(2\mathbf{i}+4\mathbf{j})+(\mathbf{i}-5\mathbf{j})]$
$= (-5\mathbf{i}-2\mathbf{j})\cdot[3\mathbf{i}-\mathbf{j}]$
$= (-5\times3)+(-2\times-1)$
$= 15+2$
$= -13$

e $(\mathbf{c}+\mathbf{a})\cdot\mathbf{b} = [(-5\mathbf{i}-2\mathbf{j})+(2\mathbf{i}+4\mathbf{j})]\cdot(\mathbf{i}-5\mathbf{j})$
$= [-3\mathbf{i}+2\mathbf{j}]\cdot(\mathbf{i}-5\mathbf{j})$
$= (-3\times1)+(2\times-5)$
$= -3-10 = -13$

2 a $\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix}-1\\0\\5\end{pmatrix}\cdot\begin{pmatrix}4\\-3\\-1\end{pmatrix} = (-1\times4)+(0\times-3)+(5\times-1)$
$= -4+0-5$
$= -9$

b $\mathbf{u}\cdot(\mathbf{v}-\mathbf{w}) = \begin{pmatrix}-1\\0\\5\end{pmatrix}\cdot\begin{pmatrix}4 & -(-1)\\-3 & -(3)\\-1 & -(-6)\end{pmatrix} = \begin{pmatrix}-1\\0\\5\end{pmatrix}\cdot\begin{pmatrix}5\\-6\\5\end{pmatrix}$
$= (-1\times5)+(0\times-6)+(5\times5)$
$= -5+0+25$
$= 20$

c $\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{w} = -9-\begin{pmatrix}-1\\0\\5\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\-6\end{pmatrix}$
$= -9-[(-1)\times(-1)+0\times3+5\times(-6)]$
$= -9-[1+0-30]$
$= -9+29 = 20$

d $2\mathbf{u}\cdot\mathbf{w} = 2(-29)$
$= -58$

e $(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{w}) = \begin{bmatrix}-1 & -4\\0 & -(-3)\\5 & -(-1)\end{bmatrix}\cdot\begin{bmatrix}-1 & +(-1)\\0 & +3\\5 & +(-6)\end{bmatrix}$
$= \begin{pmatrix}-5\\3\\6\end{pmatrix}\cdot\begin{pmatrix}-2\\3\\-1\end{pmatrix}$
$= 10+9-6 = 13$

3 a $\mathbf{a}\cdot\mathbf{b} = (2\times4)+(4\times-2)$
$= 8-8$
$= 0 \Rightarrow$ perpendicular.

b $\mathbf{c}\cdot\mathbf{d} = \begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix} = (2\times1)+(1\times2) = 4$

$|\mathbf{c}| = \sqrt{2^2+1^2} = \sqrt{5}$ $|\mathbf{d}| = \sqrt{5}$

$|\mathbf{c}|\,|\mathbf{d}| = \left(\sqrt{5}\right)^2 = 5 \neq \mathbf{c}\cdot\mathbf{d}$

So neither parallel, nor perpendicular.

c $\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix}-8\\2\\2\end{pmatrix}\cdot\begin{pmatrix}4\\-1\\-1\end{pmatrix} = (-8\times4)+(2\times-1)+(2\times-1)$
$= -32-2-2 = -36$

$|\mathbf{u}| = \sqrt{8^2+2^2+2^2} = \sqrt{64+4+4} = \sqrt{72}$

$|\mathbf{v}| = \sqrt{4^2+1^2+1^2} = \sqrt{16+1+1} = \sqrt{18}$

$|\mathbf{u}||\mathbf{v}| = \sqrt{18\times72} = 36 = -\mathbf{u}\cdot\mathbf{v}$

$\Rightarrow$ parallel.

d $\mathbf{a} = 3\mathbf{i}-2\mathbf{j}+\mathbf{k}$ $\mathbf{b} = 3\mathbf{i}-2\mathbf{j}-\mathbf{k}$

$\mathbf{a}\cdot\mathbf{b} = (3\times3)+(-2\times-2)+(1\times-1)$
$= 9+4-1$
$= 12$

$|\mathbf{a}| = \sqrt{3^2+2^2+1^2} = \sqrt{9+4+1} = \sqrt{14}$

$|\mathbf{b}| = \sqrt{3^2+2^2+1^2} = \sqrt{9+4+1} = \sqrt{14}$

$|\mathbf{a}||\mathbf{b}| = \left(\sqrt{14}\right)^2 = 14 \neq \mathbf{a}\cdot\mathbf{b}$

$\Rightarrow$ neither parallel, nor perpendicular

e $\overline{OX}\cdot\overline{OZ} = \begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix} = (1\times0)+(0\times0)+(0\times1) = 0$

$\Rightarrow$ perpendicular

f $\mathbf{n}\cdot\mathbf{m} = (2\mathbf{i}-8\mathbf{j})(-\mathbf{i}+4\mathbf{j})$
$= (2\times-1)+(-8\times4) = -2-32 = -34$

$|\mathbf{n}| = \sqrt{2^2+8^2} = \sqrt{4+64} = \sqrt{68}$

$|\mathbf{m}| = \sqrt{1^2+4^2} = \sqrt{1+16} = \sqrt{17}$

$|\mathbf{n}||\mathbf{m}| = \sqrt{17 \times 68} = 34 = -\mathbf{n} \cdot \mathbf{m}$

$\Rightarrow$ parallel.

g $\overrightarrow{AB} \cdot \overrightarrow{CD} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} = (2 \times -1) + (2 \times -1)$

$= -2 - 2$

$= -4$

$|\overrightarrow{AB}| = \sqrt{2^2 + 2^2} = \sqrt{8}$

$|\overrightarrow{CD}| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$|\overrightarrow{AB}| \cdot |\overrightarrow{CD}| = \sqrt{2 \times 8} = \sqrt{16} = 4 = -\overrightarrow{AB} \cdot \overrightarrow{CD}$

$\Rightarrow$ parallel vectors

4 $\mathbf{a} + 3\mathbf{b} = (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) + 3(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

$= (1 + 3 \times 3)\mathbf{i} + (1 + 3 \times 2)\mathbf{j} + (2 + 3 \times -1)\mathbf{k}$

$= 10\mathbf{i} + 7\mathbf{j} - \mathbf{k}$

$2\mathbf{a} - \mathbf{b} = 2(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) - (3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

$= (2(1) - 3)\mathbf{i} + (2(1) - 2)\mathbf{j} + (2(2) - (-1))\mathbf{k}$

$= -\mathbf{i} + 5\mathbf{k}$

$(\mathbf{a} + 3\mathbf{b}) \cdot (2\mathbf{a} - \mathbf{b}) = (10\mathbf{i} + 7\mathbf{j} - \mathbf{k}) \cdot (-\mathbf{i} + 5\mathbf{k})$

$= (10 \times -1) + (-1 \times 5)$

$= -10 - 5 = -15$

5 Let $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$

$\mathbf{a} \cdot \mathbf{d} = 3d_1 + (-5)d_3 = -9$

$\mathbf{b} \cdot \mathbf{d} = 2d_1 + 7d_2 = 11$

$\mathbf{c} \cdot \mathbf{d} = d_1 + d_2 + d_3 = 6$

using GDC, $d_1 = 2$, $d_2 = 1$, $d_3 = 3$

So, $\mathbf{d} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

6 $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos\theta$

$\sqrt{6} = 2\sqrt{3}\cos\theta$

$\cos\theta = \frac{\sqrt{6}}{2\sqrt{3}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45°$

7 a $\begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \end{pmatrix} = 2 \times 2 + -1 \times 5 = 4 - 5 = -1$

$\left|\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$\left|\begin{pmatrix} 2 \\ 5 \end{pmatrix}\right| = \sqrt{2^2 + 5^2} = \sqrt{4 + 25} = \sqrt{29}$

$-1 = \sqrt{5}\sqrt{29}\cos\theta$

$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{145}}\right) = 94.8°$

b $\begin{pmatrix} 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix} = 4 \times -3 + 0 \times 1 = -12$

$\left|\begin{pmatrix} 4 \\ 0 \end{pmatrix}\right| = \sqrt{4^2} = 4$

$\left|\begin{pmatrix} -3 \\ 1 \end{pmatrix}\right| = \sqrt{3^2 + 1^2} = \sqrt{10}$

$-12 = 4\sqrt{10}\cos\theta$

$\theta = \cos^{-1}\left(\frac{-3}{\sqrt{10}}\right) = 161.6°$

c $(2\mathbf{i} + 5\mathbf{j}) \cdot (2\mathbf{i} - 5\mathbf{j}) = (2 \times 2) + (5 \times -5)$

$= 4 - 25$

$= -21$

$|(2\mathbf{i} + 5\mathbf{j})| = \sqrt{2^2 + 5^2} = \sqrt{29} = |(2\mathbf{i} - 5\mathbf{j})|$

$\therefore -21 = 29\cos\theta$

$\theta = \cos^{-1}\left(\frac{-21}{29}\right) = 136.4°$

8 a $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 9 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

b $\overrightarrow{AB} \cdot \overrightarrow{AC} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = (-1 \times 1) + (5 \times -2)$

$= -1 - 10 = -11$

c $|\overrightarrow{AB}| = \sqrt{1^2 + 5^2} = \sqrt{26}$

$|\overrightarrow{AC}| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}||\overrightarrow{AC}|\cos\theta$

$-11 = \sqrt{5 \times 26}\cos\theta$

$\cos\theta = \frac{-11}{\sqrt{130}}$

9 a $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = -2 - 6 + 12 = 4$

$\left|\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}\right| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$

$\left|\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}\right| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{4 + 9 + 36} = 7$

so $4 = 3 \times 7\cos\theta$

$\cos\theta = \frac{4}{21}, \theta = 79°$

b $\begin{pmatrix}2\\3\\1\end{pmatrix}\cdot\begin{pmatrix}4\\-2\\-2\end{pmatrix}=8-6-2=0\Rightarrow$ perpendicular vectors

$\theta=90°$

c $(2\mathbf{i}-7\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}-\mathbf{k})$

$=(2\times1)+(-7\times1)+(1\times-1)=-6$

$|(2\mathbf{i}-7\mathbf{j}+\mathbf{k})|=\sqrt{2^2+7^2+1^2}=\sqrt{54}$

$|(\mathbf{i}+\mathbf{j}-\mathbf{k})|=\sqrt{1+1+1}=\sqrt{3}$

so $-6=\sqrt{162}\cos\theta$

$\theta=\cos^{-1}\left(\frac{-6}{\sqrt{162}}\right)=118.1°$

10 a $\overrightarrow{AB}=\begin{pmatrix}2\\3\\4\end{pmatrix}-\begin{pmatrix}1\\-1\\4\end{pmatrix}=\begin{pmatrix}1\\4\\0\end{pmatrix}$

$\overrightarrow{AB}==\sqrt{1^2+4^2}=\sqrt{17}$

$\overrightarrow{AC}=\begin{pmatrix}2\\-1\\-1\end{pmatrix}-\begin{pmatrix}1\\-1\\4\end{pmatrix}=\begin{pmatrix}1\\0\\-5\end{pmatrix}$

$\overrightarrow{AC}=\sqrt{1^2+(-5)^2}=\sqrt{26}$

so $AB=\sqrt{17},\ AC=\sqrt{26}$

b $\overrightarrow{AB}\cdot\overrightarrow{AC}=\begin{pmatrix}1\\4\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-5\end{pmatrix}=1$

$\overrightarrow{AB}\cdot\overrightarrow{AC}=|\overrightarrow{AB}||\overrightarrow{AC}|\cos\theta$

$1=\sqrt{17}\sqrt{26}\cos\theta$

$\frac{1}{\sqrt{442}}=\cos\theta$

c Area $ABC=\frac{1}{2}|\overrightarrow{AB}||\overrightarrow{AC}|\sin B\hat{A}C$

$=\frac{1}{2}\sqrt{442}\sin\left(\cos^{-1}\frac{1}{\sqrt{442}}\right)=10.5\text{cm}^2$

11 The x-axis has unit direction vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$

so $\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=1$

$\left|\begin{pmatrix}1\\0\\0\end{pmatrix}\right|=\sqrt{1^2}=1$ and $\left|\begin{pmatrix}1\\1\\1\end{pmatrix}\right|=\sqrt{1+1+1}=\sqrt{3}$

so $1=\sqrt{3}\cos\theta$

$\theta=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)=54.7°$

12 a $\overrightarrow{OA}=4\mathbf{i}+4\mathbf{j}-4\mathbf{k},\ \overrightarrow{OB}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$

$(\overrightarrow{OA}).(\overrightarrow{OB})=(4\times1)+(4\times2)+(-4\times3)$

$=4+8-12=0$

$=\overrightarrow{OA}$ and $\overrightarrow{OB}$ are perpendicular

b $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$=(1-4)\mathbf{i}+(2-4)\mathbf{j}+(3-(-4))\mathbf{k}$

$=-3\mathbf{i}-2\mathbf{j}+7\mathbf{k}$

$|\overrightarrow{AB}|=\sqrt{(-3)^2+(-2)^2+7^2}=\sqrt{62}\approx7.87$

13 $(2\mathbf{i}+\lambda\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}-2\mathbf{j}+3\mathbf{k})=(2\times1)+(\lambda\times-2)+(3\times1)$

$=2-2\lambda+3=0$ for perpendicular vectors.

$5=2\lambda$

$\lambda=\frac{5}{2}$

14 $\mathbf{a}+\mathbf{b}=(5+1)\mathbf{i}+(-3+1)\mathbf{j}+(7+\lambda)\mathbf{k}$

$=6\mathbf{i}-2\mathbf{j}+(7+\lambda)\mathbf{k}$

$\mathbf{a}-\mathbf{b}=(5-1)\mathbf{i}+(-3-1)\mathbf{j}+(7-\lambda)\mathbf{k}$

$=4\mathbf{i}-4\mathbf{j}+(7-\lambda)\mathbf{k}$

Now $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=(6\times4)+(-2\times-4)+(\lambda+7)(7-\lambda)$

$=24+8+49-\lambda^2=0$

$\lambda^2=81$

$\lambda=\pm9$

15 $\mathbf{a}+\mathbf{b}=\begin{pmatrix}p\\2\\-p\end{pmatrix}+\begin{pmatrix}2\\-p\\-3\end{pmatrix}=\begin{pmatrix}p+2\\2-p\\-p-3\end{pmatrix}$

$\mathbf{a}-\mathbf{b}=\begin{pmatrix}p-2\\2+p\\-p+3\end{pmatrix}$

$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=\begin{pmatrix}p+2\\2-p\\-p-3\end{pmatrix}\cdot\begin{pmatrix}p-2\\2+p\\-p+3\end{pmatrix}$

$=(p^2-4)+(4-p^2)-(9-p^2)$

$=p^2-9=0$ for perpendicular vectors.

$\Rightarrow p^2=9,\ p=\pm3$

Exercise 12J

1 a $\mathbf{r}=\begin{pmatrix}-1\\2\end{pmatrix}+t\begin{pmatrix}3\\2\end{pmatrix},\ t\in\mathbb{R}.$

b $\mathbf{r}=\begin{pmatrix}-1\\0\end{pmatrix}+t\begin{pmatrix}5\\-2\end{pmatrix},\ t\in\mathbb{R}.$

12J

c $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \\ 8 \end{pmatrix}, t \in \mathbb{R}.$

d $\mathbf{r} = 2\mathbf{j} - \mathbf{k} + t(3\mathbf{i} - \mathbf{j} + \mathbf{k}), t \in \mathbb{R}.$

2 a Position vectors are $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

Line joining the 2 points has direction

$\begin{pmatrix} 3 & -4 \\ -2 & -5 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \end{pmatrix}$

Line is $\mathbf{r} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 7 \end{pmatrix}, t \in \mathbb{R}.$

or $\begin{pmatrix} 4 \\ 5 \end{pmatrix} + t\begin{pmatrix} -1 \\ -7 \end{pmatrix}$

b Position vectors $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$

Line joining 2 points has direction

$\begin{pmatrix} 5 & -4 \\ -2 & (+2) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Line is $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \end{pmatrix}, t \in \mathbb{R}.$

or $\begin{pmatrix} 4 \\ -2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

c Position vectors $\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix}$

Line joining 2 points has direction

$\begin{pmatrix} 3 & -2 \\ 5 & -(-4) \\ 2 & -5 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \\ -3 \end{pmatrix}$

Line is $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 9 \\ -3 \end{pmatrix}, t \in \mathbb{R}.$

d Position vectors $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

Line joining 2 points has direction

$\begin{pmatrix} 0 & -1 \\ 0 & -(-1) \\ 1 & -0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

Line is $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

3 a We need a vector $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ which is perpendicular to $\mathbf{a}$

$\mathbf{a} \cdot \mathbf{p} = 0 \Rightarrow \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = 3p_1 + 2p_2 = 0$

Take $p_1 = 2, p_2 = -3$

Then $\mathbf{a} \cdot \mathbf{p} = 0$, and line is

$\mathbf{r} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \end{pmatrix}, t \in \mathbb{R}.$

b Using the same technique as in part **a**, we see $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ is perpendicular $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$

Line is $\mathbf{r} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 5 \end{pmatrix}, t \in \mathbb{R}.$

c $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$

line is $\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, t \in \mathbb{R}.$

d We require $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ so that $\mathbf{p} \cdot \mathbf{a} = 0$

$\mathbf{p} \cdot \mathbf{a} = p_1 - 3p_2 + 4p_3 = 0$

Take $p_1 = 0, p_2 = 4, p_3 = 3$ for example

Then line is $\mathbf{r} = 5\mathbf{k} + t(4\mathbf{j} + 3\mathbf{k}), t \in \mathbb{R}.$

4 a We need to know if there is a value of t for which

$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

Take $t = 2$ Then $\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

so (4, 5) lies on the line.

b Is there t so that $\begin{pmatrix} 5 \\ 1 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$?

$5 + t(4) = 5$ and $1 - 3t = -2$

$t = 0$ and $t = 1 \Rightarrow$ no such t.

so (5, –2) does not lie on the line.

c Is there t so that $\begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$?

$-1 + t = -3 \Rightarrow t = -2$

$5 + 0(t) = 5 \Rightarrow t =$ anything

$-3 - 2t = 1 \Rightarrow t = -2$

so $\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ i.e. (–3, 5, 1) lies on line.

d Is there t so that

$(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = (2\mathbf{i} - \mathbf{j} - 3\mathbf{k}) + t(-2\mathbf{j} - 3\mathbf{k})$

$1 = -1 - 2t$ and $1 = -3 - 3t$

$-2 = 2t \qquad 4 = -3t$

$t = -1$ and $t = \frac{-4}{3} \Rightarrow$ no such t.
so (2,1,1) does not lie on line.

5 $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix}, t \in \mathbb{R}$

$10 = 4 + 3t \Rightarrow 6 = 3t, t = 2$

$p = 2 - 2t = 2 - 2(2) = -2$

$q = 5 + 8t = 5 + 8(2) = 21$

6 A vertical line will have direction $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

so $\mathbf{r} = \begin{pmatrix} -6 \\ 5 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}, t \in \mathbb{R}$

7 a (1) Are the 2 lines parallel?

Is there t such that $\begin{pmatrix} 2 \\ -1 \end{pmatrix} = t\begin{pmatrix} -6 \\ 3 \end{pmatrix}$

Take $t = \frac{-1}{3}$. Then $\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{-1}{3}\begin{pmatrix} -6 \\ 3 \end{pmatrix}$ so lines parallel.

(2) Are 2 lines co-incident?

Does $\begin{pmatrix} -9 \\ 10 \end{pmatrix}$ lie on $\mathbf{r}_1$?

$\begin{pmatrix} -9 \\ 10 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Take $s = -6$.

$\begin{pmatrix} -9 \\ 10 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} - 6\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

so $\begin{pmatrix} -9 \\ 10 \end{pmatrix}$ lies on $\mathbf{r}_1 \Rightarrow$ lines co-incident

b (1) Are lines parallel?

Is there t so that $\begin{pmatrix} -4 \\ 2 \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow$ No such t, so NOT parallel.

(2) Are lines perpendicular?

Take dot product of direction vectors:

$\begin{pmatrix} -4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -4 + 4 = 0 \Rightarrow$ perpendicular.

c (1) Are the lines parallel?

Is there t so that $\begin{pmatrix} 4 \\ -3 \end{pmatrix} = t\begin{pmatrix} 8 \\ -6 \end{pmatrix}$

$t = 2$ gives

$\begin{pmatrix} 4 \\ -3 \end{pmatrix} = 2\begin{pmatrix} 8 \\ -6 \end{pmatrix} \Rightarrow$ lines parallel.

(2) Are lines co-incident?

Does $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ lie on $\mathbf{r}_1$?

$\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} + s\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ No such $s \Rightarrow$

lines NOT co-incident.

d (1) Are lines parallel?

Is there t so that $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ No such t

$\Rightarrow$ NOT parallel.

(2) Are lines perpendicular?

Take dot product of direction vectors:

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 + 2 = 3 \Rightarrow$ NOT perpendicular.

e (1) Are lines parallel?

Is there t so that

$\begin{pmatrix} 4 \\ -3 \end{pmatrix} = t\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ No such $t \Rightarrow$

lines not parallel.

(2) Are lines perpendicular?

Take dot product of direction vectors:

$\begin{pmatrix} 4 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \end{pmatrix} = 16 - 9 = 7 \Rightarrow$

NOT perpendicular.

8 a $\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \left|\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}\right| \left|\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right| \cos A$

$2 + 4 = \sqrt{1^2 + 4^2}\sqrt{2^2 + 1^2 + 1^2}\cos A$

$6 = \sqrt{17}\sqrt{6}\cos A$

$A = \cos^{-1}\left(\frac{6}{\sqrt{17}\sqrt{6}}\right) = 53.6°$

b $\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = \left|\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}\right| \left|\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}\right| \cos A$

$-2 - 2 = \sqrt{2^2 + (-2)^2}\sqrt{(-1)^2 + 3^2 + 1^2}\cos A$

$-4 = \sqrt{8}\sqrt{11}\cos A$

$A = \cos^{-1}\left(\frac{-4}{\sqrt{8}\sqrt{11}}\right) = 115.2°$

9 a A has position vector $\begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix}$. We require t so that

$\begin{pmatrix} -2 \\ -3 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ Taking $t = -1$, we see:

$-2 = -1 + (-1)1$
$-3 = -1 + (-1)2$
$-4 = 2 + (-1)6$

b $\overrightarrow{AB} = \begin{pmatrix}-6\\-7\\-2\end{pmatrix} - \begin{pmatrix}-2\\-3\\-4\end{pmatrix} = \begin{pmatrix}-4\\-4\\2\end{pmatrix}$

Taking dot product,

$$\begin{pmatrix}-4\\-4\\2\end{pmatrix} \cdot \begin{pmatrix}1\\2\\6\end{pmatrix} = -4 - 8 + 12 = 0$$

$\Rightarrow \overrightarrow{AB}$ perpendicular toL_1

10 a i $\overrightarrow{OF} = 2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$

ii $\overrightarrow{AG} = -2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$

b i $|\overrightarrow{OF}| = \sqrt{2^2 + 5^2 + 3^2} = \sqrt{38}$

ii $|\overrightarrow{AG}| = \sqrt{2^2 + 5^2 + 3^2} = \sqrt{38}$

iii $\overrightarrow{OF} \cdot \overrightarrow{AG} = (2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) \cdot (-2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})$
$= 30$

c $\overrightarrow{OF} \cdot \overrightarrow{AG} = |\overrightarrow{OF}|\ |\overrightarrow{AG}| \cos\theta$

$30 = \sqrt{38}\sqrt{38}\cos\theta$

$\theta = \cos\theta^{-1}\left(\dfrac{30}{38}\right) = 7.9°$

11 a $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 7\mathbf{i} - 8\mathbf{j} + 8\mathbf{k}$

b $\cos OAB = \dfrac{\overrightarrow{AO}.\overrightarrow{AB}}{|AO||AB|}$

$= \dfrac{-1 \times 7 + (-5) \times (-8) + 2 \times 8}{\sqrt{30}\sqrt{177}}$

$= \dfrac{49}{\sqrt{30}\sqrt{177}}$

c Let $\mathbf{r}$ be the position vector for the point P.

Then $\mathbf{r} = (1 + 7\mu)\mathbf{i} + (5 - 8\mu)\mathbf{j} + (-2 + 8\mu)\mathbf{k}$
$= (\mathbf{i} + 5\mathbf{j} - 2\mathbf{k} + \mu(7\mathbf{i} - 8\mathbf{j} - 8\mathbf{k})$
$= \overrightarrow{OA} + \mu\overrightarrow{AB}$

which is the position vector of a point on the line that passes through A with direction vector $\overrightarrow{AB}$, and hence also passes through B.

d $\overrightarrow{OP}.\overrightarrow{AB} = 0$

$$\therefore \begin{pmatrix}1 + 7\mu\\5 - 8\mu\\-2 + 8\mu\end{pmatrix} \cdot \begin{pmatrix}7\\-8\\8\end{pmatrix} = 0$$

$7 + 49\mu - 40 + 64\mu - 16 + 64\mu = 0$

$\mu = \dfrac{49}{177}$

e Use the value of μ from part **d** to get:

$$P = \frac{1}{177}\begin{pmatrix}520\\493\\38\end{pmatrix}$$

Exercise 12K

1 Equating components of $\mathbf{r}_1$ & $\mathbf{r}_2$:

$4 + 2\lambda = 11 + \mu$ (1)
$2 - 4\lambda = 16 + 2\mu$ (2)
$(1) \Rightarrow \mu = -7 + 2\lambda$
$(2) \Rightarrow 2 - 4\lambda = 16 + 2(-7 + 2\lambda)$
$2 - 4\lambda = 2 + 4\lambda$
$\lambda = 0$
$(1) \Rightarrow \mu = -7$
so intercept at (4,2)

plug μ or λ back in to get (4,2)

2 Equating components:

$4 + 8s = 6 + 9t$ (1)
$-2 + 2s = -3 + 6t$ (2)
$(1) \Rightarrow 8s = 2 + 9t$
$s = \frac{1}{8}(2 + 9t)$
$(2) \Rightarrow -2 + \frac{1}{4}(2 + 9t) = -3 + 6t$
$-8 + 2 + 9t = -12 + 24t$
$6 = 15t$
$t = \frac{6}{15}$
$(1) \Rightarrow s = \frac{1}{8}\left(2 + \frac{54}{15}\right) = \frac{7}{10}$

intersec at $\begin{pmatrix}4\\-2\end{pmatrix} + \frac{7}{10}\begin{pmatrix}8\\2\end{pmatrix} = \begin{pmatrix}\frac{48}{5}\\\frac{-3}{5}\end{pmatrix} = \frac{1}{5}\begin{pmatrix}48\\-3\end{pmatrix}$

3 $5 + 2t = 3 + 2s$ (1)
$-1 + t = -2 + s$ (2)
$2 - t = -4 + 2s$ (3)
$(2) \Rightarrow s = 1 + t$
$(1) \Rightarrow 5 + 2t = 3 + 2(1 + t)$
$5 + 2t = 5 + 2t$ (so (1) & (2) are consistent)
$(3)\ 2 - t = -4 + 2(1 + t)$
$2 - t = -2 + 2t$
$4 = 3t$
$t = \frac{4}{3}$
$(2) \Rightarrow s = 1 + \frac{4}{3} = \frac{7}{3}$
Thus l_1 & l_2 intersect.

$$\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} + \frac{7}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 23 \\ 1 \\ 2 \end{pmatrix} \text{ at } \left(\frac{23}{3}, \frac{1}{3}, \frac{2}{3}\right)$$

4 $1 + 3t = -1$ (1)

$1 - t = s$ (2)

(1) $\Rightarrow 3t = -2$

$t = \frac{-2}{3}$

(2) $\Rightarrow s = 1 - \left(\frac{-2}{3}\right) = \frac{5}{3}$

plug back in to get pt

Intersect at $\mathbf{i} + \frac{5}{3}\mathbf{j}$, i.e. at $\left(-1, \frac{5}{3}\right)$

5 If lines intersect then there are s & t so that

$3 - t = 1 + s$ (1)

$t = 4 + s$ (2)

$5 + 2t = s$ (3)

sub (2) into (3): $s = 5 + 2(4 + s)$, $s = 13 + 2s$

$s = -13$

in (2) $\Rightarrow t = 4 - 13 = -9$

check in (1): $3 - (-9) \neq 1 - 13$

$12 \neq -12$

so there are no such s & $t \Rightarrow$ skew

not parallel since $\left(\frac{-1}{2}\right)$ isn't a multiple

6 a $3 - s = 14 + 3t$ (1)

$-2 + 3s = -20 - 4t$ (2)

$5 - 5s = 6 - 3t$ (3)

(1) $\Rightarrow s = -11 - 3t$

(2) $\Rightarrow -2 - 3(11 + 3t) = -20 - 4t$

$-35 - 9t = -20 - 4t$

$-15 = 5t$

$t = -3$

(1) $\Rightarrow s = -11 - 3(-3) = -2$

check in (3): $5 - 5(-2) = 6 - 3(-3)$

$15 = 15$ so lines intersect.

Point of intersection $= 14\mathbf{i} - 20\mathbf{j} + 6\mathbf{k} - 3(3\mathbf{i} - 4\mathbf{j} - 3\mathbf{k})$ when $t = -3$

$= 5\mathbf{i} - 8\mathbf{j} + 15\mathbf{k}$

b Take dot product of direction vectors:

$(-\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}) \cdot (3\mathbf{i} - 4\mathbf{j} - 3\mathbf{k})$

$= (-1 \times 3) + (3 \times -4) + (-5 \times -3)$

$= -3 - 12 + 15$

$= 0$

$\Rightarrow$ perpendicular.

7 a $\begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ a \end{pmatrix}$

$6 + t = 5 \Rightarrow t = -1$

$3 - 2(-1) = a \Rightarrow a = 5.$

$\begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} b \\ 13 \\ -1 \end{pmatrix}$

$9 + 2t = 13 \Rightarrow t = 2.$

$6 + 2 = b \Rightarrow b = 8.$

b OP has position vector $\begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ for some t

$\overline{AB} = \begin{pmatrix} 8 \\ 13 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix}$

$(\overline{OP}) \cdot (\overline{AB}) = 0 \Rightarrow \begin{pmatrix} 6+t \\ 9+2t \\ 3-2t \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix} = 0$

$3(6 + t) + 6(9 + 2t) - 6(3 - 2t) = 0$

$18 + 3t + 54 + 12t - 18 + 12t = 0$

$27t + 54 = 0$

$t = -2$

so $\overline{OP} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}$, P is (4, 5, 7)

c $|\overline{OP}| = \sqrt{4^2 + 5^2 + 7^2} = 3\sqrt{10}$

8 a

$\overline{AB} = \mathbf{b} - \mathbf{a} = (3 - 2)\mathbf{i} + (-2 - (-1))\mathbf{j} + (-1 - 2)\mathbf{k}$

$= \mathbf{i} - \mathbf{j} - 3\mathbf{k}$

line is $(2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} - 3\mathbf{k})$ for $\lambda \in \mathbb{R}$

b $2 + \lambda = 7 + 2s$ (1)

$-1 - \lambda = s$ (2)

$2 - 3\lambda = 3 + 2s$ (3)

sub (2) in (1) $\Rightarrow 2 + \lambda = 7 + 2(-1 - \lambda)$

$2 + \lambda = 5 - 2\lambda$

$3\lambda = 3 \Rightarrow \lambda = 1$

(2) $\Rightarrow s = -1 - 1 = -2$ $= s$

(3) $\Rightarrow 2 = -3(1) - 1 = 3 + 2(-2)$

lines intersect.

point is $(2+1)\mathbf{i}+(-1-1)\mathbf{j}+(2-3)\mathbf{k}$

$3\mathbf{i}-2\mathbf{j}-\mathbf{k}$ ie $(3, -2, -1)$

c $\mathbf{a}-\mathbf{c}=(2-3)\mathbf{i}+(-1-(-2))\,\mathbf{j}+(2-(-1))\mathbf{k}$

$=-\mathbf{i}+\mathbf{j}+3\mathbf{k}$

$|AC|=\sqrt{1+1+3^2}=\sqrt{11}$

d Take dot product of direction vectors:

$(\mathbf{i}-\mathbf{j}-3\mathbf{k}).(2\mathbf{i}+\mathbf{j}+2\mathbf{k})=2-1-6=-5$

Then $-5=\sqrt{11}\,\sqrt{9}\cos\theta$

$\theta=\cos^{-1}\left(\frac{-5}{3\sqrt{11}}\right)=-120°$ (nearest degree) negative?

Exercise 12L

1 a Position of ship relative to buoy is

$\begin{pmatrix}60\\30\end{pmatrix}-\begin{pmatrix}45\\20\end{pmatrix}=\begin{pmatrix}15\\10\end{pmatrix}$ ie 10Km North, 15Km East

b $\left|\begin{pmatrix}15\\10\end{pmatrix}\right|=\sqrt{15^2+10^2}=5\sqrt{13}$ km

2 a velocity $=\frac{\text{displacement}}{\text{time}}=\begin{pmatrix}\frac{20}{4}\\\frac{-8}{4}\end{pmatrix}=\begin{pmatrix}5\\-2\end{pmatrix}\text{ms}^{-1}$

b $s(t)=\begin{pmatrix}20\\-8\end{pmatrix}+t\begin{pmatrix}5\\-2\end{pmatrix}$

$=\begin{pmatrix}20\\-8\end{pmatrix}+6\begin{pmatrix}5\\-2\end{pmatrix}=\begin{pmatrix}50\\-20\end{pmatrix}\text{m}$

c speed $=|\text{v}(t)|=\sqrt{12^2+5^2}=13\text{ ms}^{-1}$

d $\mathbf{s}(t)=(4\mathbf{i}-\mathbf{j})+t(12\mathbf{i}-5\mathbf{j})$

$\mathbf{s}(3)=(4\mathbf{i}-\mathbf{j})+3(12\mathbf{i}-5\mathbf{j})=40\mathbf{i}-16\mathbf{j}$

distance $=\sqrt{40^2+16^2}=8\sqrt{29}$ m

e want $\begin{pmatrix}20\\-8\end{pmatrix}+t\begin{pmatrix}5\\-2\end{pmatrix}=\begin{pmatrix}4\\-1\end{pmatrix}+s\begin{pmatrix}12\\-5\end{pmatrix}$

for collision

$20+5t=4+12s$ (1)

$-8-2t=-1-5s$ (2)

$(2)\Rightarrow -7+5s=2t\Rightarrow t=\frac{1}{2}(5s-7)$

$(1)\Rightarrow 20+\frac{5}{2}(5s-7)=4+12s$

$40+25s-35=8+24s$

$s=3$

$(2)\Rightarrow t=\frac{1}{2}(5\times 3-7)=4$ t≠S

When $t=4$, LS $=\begin{pmatrix}20\\-8\end{pmatrix}+\begin{pmatrix}20\\-8\end{pmatrix}=\begin{pmatrix}40\\-16\end{pmatrix}$

When $s=3$, RS $=\begin{pmatrix}4\\-8\end{pmatrix}+3\begin{pmatrix}12\\-5\end{pmatrix}=\begin{pmatrix}40\\-16\end{pmatrix}$

Hence particles will collide.

3 a Let A's position be given by

$\mathbf{a}=(3\mathbf{i}+3\mathbf{j})+t(4\mathbf{i}+3\mathbf{j})$

Let B's position be given by

$\mathbf{b}=(4\mathbf{i}+3\mathbf{j})+s(3\mathbf{i}+3\mathbf{j})$

want to find when $\mathbf{a}=\mathbf{b}$.

$3+4t=4+3s$ (1)

$3+3t=3+3s$ (2)

$(2)\Rightarrow t=s$

$(1)\Rightarrow s=t=1$

They collide 1 hour after 3 pm, ie at 4 pm.

b Collide at $(3\mathbf{i}+3\mathbf{j})+1(4\mathbf{i}+3\mathbf{j})=7\mathbf{i}+6\mathbf{j}$

4 a $\mathbf{r}_x=\begin{pmatrix}11\\3\\-3\end{pmatrix}+t\begin{pmatrix}1\\-1\\4\end{pmatrix}$ $\mathbf{r}_y=\begin{pmatrix}1\\-7\\-2\end{pmatrix}+t\begin{pmatrix}2\\1\\9\end{pmatrix}$

$|\mathbf{V}_x|=\sqrt{1^2+(-1)^2+4^2}=\sqrt{18}=3\sqrt{2}\text{ ms}^{-1}$

$|\mathbf{V}_y|=\sqrt{2^2+1^2+9^2}=\sqrt{86}\text{ ms}^{-1}$

b Meet if $\mathbf{r}_x=\mathbf{r}_y$ at the same time

$\begin{pmatrix}11\\3\\-3\end{pmatrix}+t\begin{pmatrix}1\\-1\\4\end{pmatrix}=\begin{pmatrix}1\\-7\\-2\end{pmatrix}+s\begin{pmatrix}2\\1\\9\end{pmatrix}$

$11+t=1+2s$ (1)

$3-t=-7+s$ (2)

$-3+4t=-2+9s$ (3)

$(1)\Rightarrow t=2s-10$

$(2)\Rightarrow 3-(2s-10)=-7+s$

$13-2s=-7+s$

$20=3s,\ s=\frac{20}{3}$

$(1)\Rightarrow t=2\left(\frac{20}{3}\right)-10=\frac{10}{3}\neq s$

so ships do not collide.

c $\mathbf{r}_x(10) = \begin{pmatrix} 11 \\ 3 \\ -3 \end{pmatrix} + 10\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 21 \\ -7 \\ 37 \end{pmatrix}$

$\mathbf{r}_y(10) = \begin{pmatrix} 1 \\ -7 \\ -2 \end{pmatrix} + 10\begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix} = \begin{pmatrix} 21 \\ 3 \\ 88 \end{pmatrix}$

$\mathbf{r}_y - \mathbf{r}_x = \begin{pmatrix} 0 \\ 10 \\ 51 \end{pmatrix}$,

$|\mathbf{r}_y - \mathbf{r}_x|$

$= \sqrt{10^2 + 51^2} = \sqrt{2701} \approx 51.97\text{m}$

Review exercise

1 $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} 7 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 9 \\ -3 \\ -6 \end{pmatrix}$

Now $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 9 \\ -3 \\ -6 \end{pmatrix} = \frac{-1}{3}\overrightarrow{BC}$

Since they contain a common point (B), A, B, C are collinear.

2 The sides of the triangle are given by the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{BC}$

$\overrightarrow{AB}$ = (21 + 2J) – (5i – j + 6k)

= – 3i + 3j – 6k

$\overrightarrow{AC}$ = (–3i – 5j + 8k) – (5i – j + 6k)

= –8i – 4j + 2k

Now $\overrightarrow{AB} \cdot \overrightarrow{AC}$ = (–3i + 3j – 6k) · (–8i – 4j + 2k)

= (–3 × –8) + (3 × –4) + (–6 × 2)

= +24 – 12 – 12

= 0

Since $\overrightarrow{AB} \cdot \overrightarrow{AC} = 0$, the vectors are perpendicular. Hence, A, B, and C form a right-angled triangle.

3 $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 5+1 \\ -1+3 \\ -3+(-5) \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -8 \end{pmatrix}$

$\mathbf{a} - \mathbf{b} = \begin{pmatrix} 5-1 \\ -1-3 \\ -3-(-5) \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix}$

$(\mathbf{a} - \mathbf{b}).(\mathbf{a} + \mathbf{b}) = \begin{pmatrix} 6 \\ 2 \\ -8 \end{pmatrix} . \begin{pmatrix} 4 \\ -4 \\ 2 \end{pmatrix} = 24 - 8 - 16 = 0$

$\Rightarrow$ **a** – **b** and **a** + **b** are perpendicular.

4 We need s & t so that

$7s = 3 + 2t$ (1)

$6 + 3s = 1 + 4t$ (2)

$-1 + s = 2 - t$ (3)

(3) $\Rightarrow s = 3 - t$

(1) $\Rightarrow 7(3 - t) = 3 + 2t$

$21 - 7t = 3 + 2t$

$18 = 9t$

$t = 2$

(3) $\Rightarrow s = 3 - 2 = 1$

check in (2) $\Rightarrow 6 + 3(1) = 1 + 4(2)$

LS = RS = 9 so s and t exist.

so P has position vector $\begin{pmatrix} 0 \\ 6 \\ -1 \end{pmatrix} + \begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ 0 \end{pmatrix}$

Point (7, 9, 0)

5 a $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

b $\overrightarrow{AB} \cdot \overrightarrow{AC} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} . \begin{pmatrix} -1 \\ -2 \end{pmatrix} = -3 - 6 = -9$

c $\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}||\overrightarrow{AC}|\cos B\hat{A}C$

$-9 = \sqrt{3^2 + 3^2}\sqrt{(-1)^2 + (-2)^2}\cos B\hat{A}C$

$-9 = \sqrt{18}\sqrt{5}\cos B\hat{A}C$

$-9 = 3\sqrt{2}\sqrt{5}\cos B\hat{A}C$

$\cos B\hat{A}C = \frac{-3}{\sqrt{2}\sqrt{5}}$

6 a P has position vector

$\begin{pmatrix} 6 \\ 2 \\ -3 \end{pmatrix} + 4\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6-8 \\ 2+8 \\ -3+4 \end{pmatrix} = \begin{pmatrix} -2 \\ 10 \\ 1 \end{pmatrix}$

P is (–2, 10, 1)

b Suppose $\begin{pmatrix}-2\\10\\1\end{pmatrix} = \begin{pmatrix}0\\-12\\7\end{pmatrix} + t\begin{pmatrix}-1\\11\\-3\end{pmatrix}$ for some t

$-2 = -t \quad (1)$

$10 = -12 + 11t \quad (2)$

$1 = 7 - 3t \quad (3)$

$(1) \Rightarrow t = 2$

$(2) \Rightarrow 10 = -12 + 11(2) = 10$

$(3) \Rightarrow 1 = 7 - 3(2) = 7 - 6 = 1$

so equations are consistent.

so $t = 2$ gives $\begin{pmatrix}-2\\10\\1\end{pmatrix}$ $\therefore P$ lies on L_2

7 **a** L_2: $\mathbf{r} = \begin{pmatrix}2\\2\\4\end{pmatrix} + s\begin{pmatrix}1\\3\\2\end{pmatrix}$

b
$0 = \begin{pmatrix}1\\7\\3\end{pmatrix} \cdot \begin{pmatrix}4\\x\\1\end{pmatrix} = 4 + 7x + 3 = 7 + 7x \Rightarrow x = -1$

c $\begin{pmatrix}2\\-3\\-3\end{pmatrix} + t\begin{pmatrix}1\\7\\3\end{pmatrix} = \begin{pmatrix}7\\5\\1\end{pmatrix} + q\begin{pmatrix}4\\-1\\1\end{pmatrix}$

$2 + t = 7 + 4q \quad (1)$

$-3 + 7t = 5 - q \quad (2)$

$-3 + 3t = 1 + q \quad (3)$

8 Suppose $\mathbf{r}_1 = \mathbf{r}_2$

Then $\begin{pmatrix}-4\\3\end{pmatrix} + \lambda\begin{pmatrix}4\\17\end{pmatrix} = \begin{pmatrix}4\\9\end{pmatrix} + \mu\begin{pmatrix}-12\\5\end{pmatrix}$

$-4 + 4\lambda = 4 - 12\mu \quad (1)$

$3 + 17\lambda = 9 + 5\mu \quad (2)$

$(1) \Rightarrow 4\lambda = 8 - 12\mu$

$\lambda = 2 - 3\mu$

$(2) \Rightarrow 17(2 - 3\mu) = 6 + 5\mu$

$34 - 51\mu = 6 + 5\mu$

$28 = 56\mu$

$\mu = \frac{1}{2}$

$(1) \Rightarrow \lambda = 2 - \frac{3}{2} = \frac{1}{2}$

So ships collide after $\frac{1}{2}$ hour, ie 12.30 pm.

collide at $\begin{pmatrix}-4\\3\end{pmatrix} + \frac{1}{2}\begin{pmatrix}4\\17\end{pmatrix} = \begin{pmatrix}-2\\\frac{23}{2}\end{pmatrix}$

b At 12.15, A has position $\begin{pmatrix}-4\\3\end{pmatrix} + \frac{1}{4}\begin{pmatrix}4\\17\end{pmatrix} = \begin{pmatrix}-3\\\frac{29}{4}\end{pmatrix}$

so after 12:15, A's position given by

$\mathbf{r}_1 = \begin{pmatrix}-3\\\frac{29}{4}\end{pmatrix} + t\begin{pmatrix}16\\17\end{pmatrix}$ where t is time after 12:15

At 12.30, A's position is

$\mathbf{r}_1 = \begin{pmatrix}-3\\\frac{29}{4}\end{pmatrix} + \frac{1}{4}\begin{pmatrix}16\\17\end{pmatrix} = \begin{pmatrix}1\\\frac{23}{2}\end{pmatrix}$

Distance is $\begin{pmatrix}-2\\\frac{23}{2}\end{pmatrix} - \begin{pmatrix}1\\\frac{23}{2}\end{pmatrix} = \begin{pmatrix}-3\\0\end{pmatrix}$

so ships are 3km apart.

Review exercise – GDC

1 $\begin{pmatrix}3\\5\end{pmatrix} \cdot \begin{pmatrix}2\\-4\end{pmatrix} = \left|\begin{pmatrix}3\\5\end{pmatrix}\right|\left|\begin{pmatrix}2\\-4\end{pmatrix}\right| \cos A$

$6 - 20 = \sqrt{3^2 + 5^2}\sqrt{2^2 + (-4)^2} \cos A$

$\frac{-14}{\sqrt{34}\sqrt{20}} = \cos A$

$A = 122.4 \approx 122°$

2 **a** $\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ}$

$= \begin{pmatrix}2\\-1\\5\end{pmatrix} - \begin{pmatrix}3\\-1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\5\end{pmatrix}$

$\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$

$= \begin{pmatrix}3\\-2\\1\end{pmatrix} - \begin{pmatrix}3\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\\1\end{pmatrix}$

b $\begin{pmatrix}-1\\0\\5\end{pmatrix} \cdot \begin{pmatrix}0\\-1\\1\end{pmatrix} = \left|\begin{pmatrix}-1\\0\\5\end{pmatrix}\right|\left|\begin{pmatrix}0\\-1\\1\end{pmatrix}\right| \cos P\hat{Q}R$

$0 + 0 + 5 = \sqrt{(-1)^2 + 0^2 + 5^2}$
$\sqrt{0^2 + (-1)^2 + 1^2} \cos P\hat{Q}R$

$\frac{5}{\sqrt{26}\sqrt{2}} = \cos P\hat{Q}R$

$= 46.1 \approx 46°$

c $\frac{1}{2}|\overrightarrow{QR}||\overrightarrow{QP}| \sin 46 = \text{Area}$

$= \frac{1}{2}\sqrt{52} \sin 46$

$= 2.60 \text{ units}^2$

3 a i $\overrightarrow{OC} = 4\mathbf{j}$

ii $\overrightarrow{OB} = \mathbf{i} + \sqrt{2^2 - 1^2}\,\mathbf{k} = \mathbf{i} + \sqrt{3}\,\mathbf{k}$

iii $\overrightarrow{OD} = 2\mathbf{i} + 4\mathbf{j}$

b i $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

$= -\mathbf{i} - \sqrt{3}\mathbf{k} + 4\mathbf{j} = -\mathbf{i} + 4\mathbf{j} - \sqrt{3}\mathbf{k}$

ii $\overrightarrow{BD} = \mathbf{i} + 4\mathbf{j} - \sqrt{3}\mathbf{k}$

c i $|\overrightarrow{BC}| = \sqrt{1^2 + 16 + 3} = \sqrt{20} = 2\sqrt{5}$

ii $|\overrightarrow{BD}| = \sqrt{1^2 + 16 + 3} = \sqrt{20} = 2\sqrt{5}$

iii $(-\mathbf{i} + 4\mathbf{j} - \sqrt{3}\mathbf{k})\cdot(\mathbf{i} + 4\mathbf{j} - \sqrt{3}\mathbf{k}) =$

$-1 + 16 + 3 = 18$

d. $18 = 2\sqrt{5} \times 2\sqrt{5} \cos D\hat{B}C$

$\frac{18}{20} = \frac{9}{10} \cos D\hat{B}C$

$25.8° = D\hat{B}C$

4 a If perpendicular, $\mathbf{a}\ \ \mathbf{b} = 0$

$(x\mathbf{i} + (x-2)\mathbf{j} + \mathbf{k}) \quad (x^2\mathbf{i} - 2x\mathbf{j} - 12x\mathbf{k}) = 0$

$x^3 - 2x(x-2) - 12x = 0$

$x^3 - 2x^2 + 4x - 12x = 0$

$x(x^2 - 2x - 8) = 0$

$x(x-4)(x+2) = 0$

$x = 0$, $x = 4$, or $x = -2$

b Let $x = -1$. $\mathbf{a} = -1\mathbf{i} + -3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 12\mathbf{k}$

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos C$

$-1 + -6 + 12 = \sqrt{(-1)^2 + (-3)^2 + 1^2}\ \sqrt{1^2 + 2^2 + 12^2} \cos C$

$5 = \sqrt{11}\sqrt{149} \cos C$

$\frac{5}{\sqrt{11}\sqrt{149}} = \cos C$

$82.9° = C$

5 a $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= \begin{pmatrix}1\\5\\5\end{pmatrix} - \begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{pmatrix}0\\6\\2\end{pmatrix}$

$\overrightarrow{OP}.\overrightarrow{PQ} = \begin{pmatrix}1\\-1\\3\end{pmatrix}.\begin{pmatrix}0\\6\\2\end{pmatrix} = 0 - 6 + 6 = 0$

$\therefore \overrightarrow{OP}$ is perpendicular to $\overrightarrow{PQ}$

b $\mathbf{r}_1 = \begin{pmatrix}1\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\6\\2\end{pmatrix}$

c If intersect, $r_1 = r_2$

$\begin{pmatrix}1\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\6\\2\end{pmatrix} = \begin{pmatrix}2\\-1\\2\end{pmatrix} + \mu\begin{pmatrix}1\\-3\\-2\end{pmatrix}$

$1 = 2 + \mu \qquad \mu = -1$

$-1 + 6\lambda = -1 - 3\mu \qquad 6\lambda = -3\mu = 3 \quad \lambda = \frac{1}{2}$

$3 + 2\lambda = 2 - 2\mu \quad$ LHS $3 + 2 \times \frac{1}{2} = 4$

RHS $2 - 2 \times -1 = 2 + 2 = 4$

consistent values for λ and μ in all 3 equations

lines intersect

Let $\lambda = \frac{1}{2}$ in r_1

Position vector $= \begin{pmatrix}1+0\\-1+3\\3+1\end{pmatrix} = \begin{pmatrix}1\\2\\4\end{pmatrix}$

d

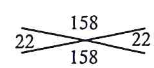
$\begin{pmatrix}0\\6\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-3\\-2\end{pmatrix} = \left|\begin{pmatrix}0\\6\\2\end{pmatrix}\right|\left|\begin{pmatrix}1\\-3\\-2\end{pmatrix}\right| \cos A$

$-18 + -4 = \sqrt{0^2 + 6^2 + 2^2}\ \sqrt{1^2 + (-3)^2 + (-2)^2} \cos A$

$-22 = \sqrt{40}\sqrt{14} \cos A$

$\frac{-22}{\sqrt{40}\sqrt{14}} = \cos A$

$A = 158°$



acute angle between lines is 22°

6 a $t = 0 \quad A = \begin{pmatrix}0\\0\\6\end{pmatrix}$

$t = 2 \quad B = \begin{pmatrix}6\\-2\\6\end{pmatrix}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \begin{pmatrix}6\\-2\\6\end{pmatrix} - \begin{pmatrix}0\\0\\6\end{pmatrix} = \begin{pmatrix}6\\-2\\0\end{pmatrix}$

b Position vector = initial position + t (directional vector of $\overrightarrow{AB}$)

$=$ initial position $+ t\begin{pmatrix}3\\-1\\0\end{pmatrix}$

c (36, 18, 0)

d

$$v = \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix}, \text{ speed} = |v| = \sqrt{(-3)^2 + (-4)^2 + 1^2}$$

$$= \sqrt{9 + 16 + 1} = \sqrt{26}$$

$$= 5.10 \text{ ms}^{-1}$$

e

$$\begin{pmatrix} 36 \\ 18 \\ 1 \end{pmatrix} + t\begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} + s\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

$36 - 3t = 3s \quad 36 - 18 = 18 \quad s = 6$

$18 - 4t = -s \quad 18 - 24 = -s \quad s = 6$ consistent

$t = 6$

$t = 6$ seconds

f $c = (36 - 18, 18 - 24, 0 + 6)$

$= (18, -6, 6)$