CHAPTER SUMMARY

Exponents and Logarithms

- Logarithms are defined as $a^x = b \Leftrightarrow x = \log_a b$.
- The following laws of logarithms are useful in simplifying logarithmic expressions and solving logarithmic equations:
  - Product law: $\log_c a + \log_c b = \log_c ab$
  - Quotient law: $\log_c a - \log_c b = \log_c \frac{a}{b}$
  - Power law: $\log_c a^r = r \log_c a$
- The change of base formula, $\log_b a = \frac{\log_c a}{\log_c b}$, is useful when simplifying a logarithmic expression.
- The argument of a logarithm must always be positive, so some logarithmic equations have extraneous solutions.

Sequences and Series

- A sequence is an ordered list of numbers that follows a pattern. Sequences can be arithmetic, geometric, or neither.
- The formula $u_n = u_1 + (n-1)d$ gives the general term in an arithmetic sequence. The formula $u_n = u_1 r^{n-1}$ gives the general term in a geometric sequence.
- A series is formed by adding up the terms in a sequence.
- The sum of the first n terms of an arithmetic series can be found using the formulas $S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$.
- The sum of the first n terms of a geometric series can be found using the formulas $S_n = \frac{u_1\left(r^n - 1\right)}{r-1} = \frac{u_1\left(1-r^n\right)}{1-r}$.
- A geometric series converges if $|r| < 1$. In this case, the sum to infinity can be found using the formula $S_\infty = \frac{u_1}{1-r}$.
- Sigma notation can be used to represent the partial sum of series. The sum of the first n terms of a series is written as $\sum_{i=1}^{n} u_i$.

A finite sum means a sum exists!

The Binomial Theorem

- The expansion of a power of a binomial is given by the following formula:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n$$

- Binomial coefficients can be found using the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.
- Binomial coefficients can also be evaluated using the combination, ${}_nC_r$, on a calculator.

IB Math SL
Algebra
Review

Solution

$$4^{x+1}=\left(\frac{1}{8}\right)^{3x}$$
$$\left(2^2\right)^{x+1}=\left(2^{-3}\right)^{3x}$$
$$2^{2(x+1)}=2^{-3(3x)}$$
$$2(x+1)=-3(3x)$$
$$2x+2=-9x$$
$$11x=-2$$
$$x=-\frac{2}{11}$$

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Exponents and Logarithms—Practice Exercises

(Answers on page 51)

1. Simplify each exponential expression to the form x^n where n is positive.

   (a) $\sqrt{\dfrac{x^3\cdot x^8}{x^6}}$

   (b) $\left(\dfrac{x^{-2}}{x^{-6}}\right)^3$

   (c) $\left(\dfrac{1}{x^2}\right)^{-3}$

2. (No Calculator Allowed) Write each expression as a single logarithm.

   (a) $\log 8+\log 6-\log 12$

   (b) $2\log 6-\frac{1}{2}\log 9$

   (c) $-3\log 2+2\log 4$

3. (No Calculator Allowed) Let $x=\log_A 5$, $y=\log_A 2$, and $z=\log_A 3$. Write each expression in terms of x, y, and/or z.

   (a) $\log_A 12$

   (b) $\log_A 7.5$

   (c) $\log_5 3$

4. (No Calculator Allowed) Solve for x: $27^{x-2}=\left(\frac{1}{9}\right)^{3x}$

5. Solve each logarithmic equation. Remember to check for extraneous solutions.

   (a) $5\log_3 x-2=8$

   (b) $5\log_3(x-2)=8$

   (c) $\ln(5x)+\ln(2x)=3$

   (d) $\log(5x+1)-\log(x-2)=1$

   (e) $\log_2(x-5)+\log_2(x+1)=4$

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6. Solve each exponential equation. Round answers to three significant figures.

   (a) $3^{2x+1} = 5$
   (b) $2e^{3x} + 1 = 9$
   (c) $2^{3x-2} = 64$

7. (No Calculator Allowed) Let $f(x) = 4(2)^{x-1} + 3$

   (a) Find the coordinates of the y-intercept on the graph of f.
   (b) The graph of f passes through the point $(k, 35)$. Find the value of k.

FEATURED QUESTION

May 2011, Paper 2

Let $f(x) = \log_3 \frac{x}{2} + \log_3 16 - \log_3 4$, for $x > 0$.

(a) Show that $f(x) = \log_3 2x$.

(b) Find the value of $f(0.5)$ and of $f(4.5)$.

The function f can also be written in the form $f(x) = \dfrac{\ln ax}{\ln b}$.

(c) (i) Write down the value of a and of b.
    (ii) Hence on graph paper, sketch the graph of f for $-5 \le x \le 5$, $-5 \le y \le 5$, using a scale of 1 cm to 1 unit on each axis.
    (iii) Write down the equation of the asymptote.

(d) Write down the value of $f^{-1}(0)$.

The point A lies on the graph of f. At A, $x = 4.5$.

(e) On your diagram, sketch the graph of f^{-1}, noting clearly the image of point A.

(Answer on page 62)

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Sequences and Series—Practice Exercises

(Answers on page 54)

1. Classify each sequence as arithmetic or geometric. Then find an expression for the general term of each sequence. Finally, find the 10th term in the sequence.

   (a) 3, 7, 11, 15, . . .
   (b) 6.2, 8, 9.8, 11.6, . . .
   (c) $\frac{1}{3}$, 1, 3, 9, . . .
   (d) 23, 21, 19, 17, . . .
   (e) 8, 12, 18, 27, . . .
   (f) 960, 288, 86.4, 25.92, . . .

2. Find the sum of the first 15 terms of each series.

   (a) $\frac{1}{2} + 2 + \frac{7}{2} + 5 + \ldots$
   (b) $2 + 4 + 8 + 16 + \ldots$
   (c) $15 + 11 + 7 + 3 + \ldots$

3. Find the sum of the given series. (Note that you must first calculate the number of terms.)

   (a) $3.7 + 4.2 + 4.7 + \ldots + 27.2$
   (b) $80 + 74 + 68 + \ldots + 14$
   (c) $0.125 + 0.25 + 0.5 + \ldots + 64$

4. In an arithmetic sequence, $u_{15} = 25.4$ and $u_{28} = 46.2$.

   (a) Find an expression for the general term.
   (b) Find the first term to exceed 100.

5. (No Calculator Allowed) Consider two geometric series given by $\sum_{r=1}^{n} 6\cdot\left(\frac{1}{2}\right)^r$ and $\sum_{r=1}^{n} 6\cdot\left(\frac{3}{2}\right)^r$.

   (a) Write down the first four terms of each series. Simplify your answers, but leave them exact.
   (b) One of these series can be summed to infinity. Find this infinite sum.

6. The current population of a city is 275,000 people, and the population is growing at a constant rate of 3.1% annually.

   (a) Write an expression for the population at the end of n full years.
   (b) How many full years are required for the population to double?

7. (No Calculator Allowed) The first three terms of an arithmetic sequence are $2x-5$, $x-1$, and $2x$. Find the value of x.

FEATURED QUESTION

May 2006, Paper 2

Consider the geometric sequence $-3, 6, -12, 24, \ldots$

(a) (i) Write down the common ratio.
    (ii) Find the 15th term.

Consider the sequence $x-3, x+1, 2x+8, \ldots$

(b) When $x = 5$, the sequence is geometric.
    (i) Write down the first three terms.
    (ii) Find the common ratio.

(c) Find the other value of x for which the sequence is geometric.

(d) For this value of x, find
    (i) The common ratio;
    (ii) The sum of the infinite sequence.

(Answer on page 64)

EXAMPLE

Find the term in x^{10} in the expansion of $(3x^2-1)^8$.

Solution

$n=8$, $a=3x^2$, $b=-1$, $r=?$

Since the term number is not given in this problem, plug in the other three values first and then determine the value of r.

$$\binom{8}{r}\cdot\left(3x^2\right)^{8-r}\cdot(-1)^r=\binom{8}{r}\cdot 3^{8-r}\cdot x^{2(8-r)}\cdot(-1)^r$$

We need x^{10}, so $2(8-r)=10$. Solving this equation gives $r=3$.

$$\binom{8}{3}\cdot 3^5\cdot x^{10}\cdot(-1)^3=-13{,}608x^{10}$$

The Binomial Theorem – Practice Exercises

(Answers on page 61)

1. Write out the expansion of each binomial. Simplify your answers.

   (a) $(x-3)^4$
   (b) $(2x-1)^5$
   (c) $(p+q)^7$
   (d) $(x-h)^4$

2. Consider the expression $(2x-5)^8$.

   (a) Write down the number of terms in the expansion of this expression.
   (b) Find the 6th term in the expansion of this expression.

3. (No Calculator Allowed) Find the term containing p^7 in the expansion of $(p+q)^{10}$.

4. Find the values of p and q in the binomial expansion shown here:
   $(3x+p)^6=729x^6-486x^5+qx^4+\ldots$

5. ~~(No Calculator Allowed) Find the constant term in the expansion of $(3x+2)^2\left(1+\frac{1}{x}\right)^4$.~~

Additional Problem:

5. Find the term in x^3 in $(3x+4)(2x-1)^5$
   (Hint ... Use #1(b) above to help)