Scalar Product and Angles Between Vectors

Definitions: If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ then the scalar product (also known as the dot product) can be found by multiplying the coefficients of i together and the coefficients of j together and adding them up.

Example: If $\mathbf{u} = 8\mathbf{i} - 2\mathbf{j}$ and $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$, find the scalar product.

$$\mathbf{u}\bullet\mathbf{v} = (8)(4) + (-2)(-3) = 32 + 6 = 38$$

Using the Cosine Rule (also known as Law of Cosine) another rule is derived where $\mathbf{a}\bullet\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \Theta$ where Θ corresponds to the angle between the two vectors.

If you don't know then angle between two vectors, you could use the above equation or solve for cosine…

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Example: Find the measure of the angle between the 2 vectors $\mathbf{u} = -6\mathbf{i} - 3\mathbf{j}$ and $\mathbf{v} = -8\mathbf{i} + 4\mathbf{j}$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{(-6)(-8) + (-3)(4)}{\sqrt{(-6)^2 + (-3)^2}\cdot\sqrt{(-8)^2 + (4)^2}} = \frac{36}{\sqrt{45}\cdot\sqrt{80}} = \frac{36}{\sqrt{3600}} = \frac{36}{60} = \frac{3}{5}$$

$$\cos\theta = \frac{3}{5}$$

$$\cos^{-1}\left(\frac{3}{5}\right) = 53.13°$$

The formula derived from the cosine rule can also be used to determine whether two vectors are perpendicular (also known as orthogonal).

Perpendicular: Two vectors **a** and **b** that are perpendicular have an angle in between of 90°. Thus they have a scalar product of 0.

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos 90°$$
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| (0)$$
$$\mathbf{a} \cdot \mathbf{b} = 0$$

Example: Determine whether **u** and **v** are perpendicular, parallel, or neither if $\mathbf{u} = 0.25(3\mathbf{i} - \mathbf{j})$ and $\mathbf{v} = 5\mathbf{i} + 6\mathbf{j}$

First, $\mathbf{u} = 0.75\mathbf{i} - 0.25\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} + 6\mathbf{j}$

$$\mathbf{u} \cdot \mathbf{v} = (0.75)(5) + (-0.25)(6) = 3.75 + -1.5 = 2.25$$

Since the scalar product isn't 0 and **v** isn't a scalar multiple of **u** they are neither.

Example: Find the value of k such that **u** and **v** are perpendicular if $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} - k\mathbf{j}$

Since the vectors are perpendicular, the angle in between is 90° so as discussed earlier, they should have a scalar product of 0.

$$\mathbf{u} \cdot \mathbf{v} = (3)(2) + (2)(-k) = 6 + -2k$$

$$6 + -2k = 0$$

$$k = 3$$