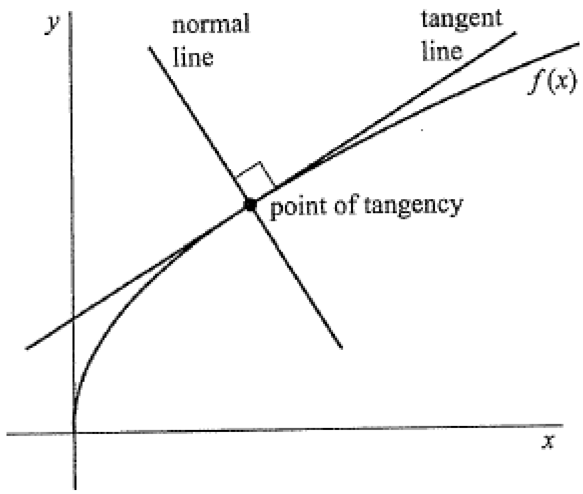
Applications of the Derivative



To find the equation of the **tangent** line to $f(x)$ at a given x-value, follow these steps…

- Find the y-coordinate of the point of tangency by plugging x into the function $f(x)$
- Find the derivative $f'(x)$
- Find the slope of the tangent (m) by plugging x into the derivative $f'(x)$
- Substitute the slope and the coordinate of the point of tangency into the point-slope equation of a line, $y - y_1 = m(x - x_1)$
- Simplify the equation to $y = mx + b$ form.

To find the equation of the **normal** line to $f(x)$ at a given x-value, follow the same steps as above but substitute in the **perpendicular** (opposite reciprocal) slope instead.

Example: Find the equation of the tangent and the normal to $f(x) = 3x^2 - 5x + 3$ at the point where $x = 2$.

$$y = f(2) = 3(2)^2 - 5(2) + 3 = 5$$

$$f'(x) = 6x - 5 \qquad m = f'(2) = 6(2) - 5 = 7 \qquad m_n = -\frac{1}{7}$$

Tangent line:
$$\begin{aligned} y - 5 &= 7(x-2) \\ y - 5 &= 7x - 14 \\ y &= 7x - 9 \end{aligned}$$

Normal line:
$$\begin{aligned} y - 5 &= -\tfrac{1}{7}(x-2) \\ y - 5 &= -\tfrac{1}{7}x + \tfrac{2}{7} \\ y &= -\tfrac{1}{7}x + \tfrac{37}{7} \end{aligned}$$

Graphing

A function is **increasing** when the function has a positive slope. Similarly, a function is **decreasing** when the function has a negative slope.

A function has a **relative maximum** (or local) when the function changes from increasing to decreasing. A function has a **relative minimum** (or local) when the function changes from decreasing to increasing. These are called **relative extrema**

When a function curves upward like a smile it is called **concave up** and when it curves downward like a frown it is called **concave down.** Points where it change concavity are called **inflection points**.

A **stationary point** is where $f'(x) = 0$. A **critical number** of f is a point where $f'(x) = 0$ OR $f'(x)$ is undefined.

If $f'(x) > 0$ for all x in (a,b), then f is **increasing** on (a,b). If $f'(x) < 0$ for all x in (a,b), then f is **decreasing** on (a,b).

FIRST DERIVATIVE TEST

Definition: A function has a **relative maximum** (or local) when the function changes from increasing to decreasing. Hence, if $f'(x)$ changes from positive to negative at x = c, then f has a relative maximum at $(c, f(c))$.

Definition: A function has a **relative minimum** (or local) when the function changes from decreasing to increasing. Hence, if $f'(x)$ changes from negative to positive at x = c, then f has a relative minimum at $(c, f(c))$.

Concavity TEST

Definition: If $f''(x) > 0$ for all x in (a,b) then f is concave up on (a,b).

Definition: If $f''(x) < 0$ for all x in (a,b), then f is concave down on (a,b).

Definition: A point on the graph of f is an **inflexion point** if $f''(x) = 0$ AND if f'' changes signs.

Example: Use the derivative of $f(x) = 2x^3 - 3x^2 - 12x$ to find the intervals on which f is increasing or decreasing. Then find any relative minimums or maximums. Then use the second derivative to find the intervals where the function is concave up and concave down. Also find the inflexion points. Then use that info and intercepts to sketch a graph.

$f'(x) = 6x^2 - 6x - 12$

$6(x^2 - x - 2) = 0$

$(x-2)(x+1) = 0$

$x = -1, 2$

$+ \quad | \quad - \quad | \quad +$

$\quad -1 \qquad 2$

(plug #'s into $f'(x)$)

increasing $(-\infty,-1) \cup (2,\infty)$

decreasing $(-1,2)$

$+ \quad | \quad - \quad | \quad +$

$\quad -1 \qquad 2$

max @ $x=-1$

min @ $x=2$

$f(-1) = 7 \quad f(2) = -20$

relative maximum$(-1,7)$ relative minimum$(2,-20)$

$f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6$

$12x - 6 = 0$

$x = 1/2$

$- \quad | \quad +$

$\quad 1/2$ plug into $f''(x)$

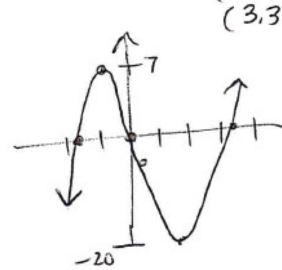
CC↑ $(1/2, \infty)$

CC↓ $(-\infty, 1/2)$

inf. pt $(1/2, f(1/2))$ $(1/2, -6.5)$

x-int: $(0,0)$ $\left(\frac{3 \pm \sqrt{108}}{4}, 0\right)$ → $(-1.8, 0)$ $(3.31, 0)$

y-int: → $(0,0)$



Optimization

Word problems involve finding the maximum or minimum values such as maximizing area or minimizing cost are known as **optimization problems**.

Tips for solving optimization problems:

1. DRAW A PICTURE!
2. Assign your variables
3. Write an equation to be **optimized** in terms of two variables
4. Find values that are *sensible* for the problem where the derivative equals zero
5. Verify that you have the desired max or min using the first or second derivative test.

*Remember to check the endpoints on a closed interval!

Example: A rectangular plot of farmland is enclosed by 180 m or fencing material on three sides. The fourth side of the plot is bounded by a stone wall. Find the dimensions of the plot that enclose the maximum area. Find the max area.

$A = xy$

$180 = 2x + y$

$180 - 2x = y$

$A = (180 - 2x)x$

$= 180x - 2x^2$

$A' = 180 - 4x$

$0 = 180 - 4x \quad x = 45\text{ m}$

$y = 90\text{ m}$

Area $= 45 \cdot 90 = 4050\text{ m}^2$

Motion

Definition: If an object is moving along a straight line, its position from an origin at any time *t* can be modeled by s(t), called the **displacement function**. (In Calculus, we called this the position function).

Definition: The **initial position** is the position when t = 0, hence, s(0).

The instantaneous rate of change of displacement is the **velocity function.** v(t) = s '(t).
When v(t) > 0, the object is moving to the right (or up).
When v(t) < 0, the object is moving to the left (or down).
When v(t) = 0, the object is at rest. It is also at its minimum or maximum height and is changing direction.

***Speed** is the absolute value of velocity.

As previously stated, the instantaneous rate of change of displacement is the **acceleration function.** a(t) = s' '(t).
When a(t) > 0, the velocity of the object is increasing.
When a(t) < 0, the velocity of the object is decreasing.
When a(t) = 0, the velocity is constant. It is also at its minimum or maximum velocity.

When velocity and acceleration have the same sign, the object in motion is speeding up
When velocity and acceleration have different signs, the object in motion is slowing down

Example: A particle moves in a straight line with a displacement of s meters t seconds after leaving a fixed point. The displacement function is given by s(t) = $2t^3 - 21t^2 + 60t + 3$, for $t \geq 0$.

a.) Find the velocity of the particle at any time t.

$v(t) = s'(t) = 6t^2 - 42t + 60$

b.) Find the initial position and initial velocity of the particle.

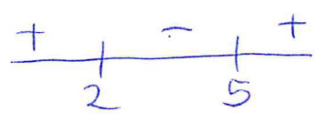
$s(0) = 3m$ $\quad$ $v(0) = s'(0) = 60\ m/s$

c.) Find when the particle is at rest.

$v(t) = 0 \quad 6t^2 - 42t + 60 = 0$ $\quad$ $t^2 - 7t + 10 = 0$
$(t-5)(t-2) = 0$
$t = 2$ and 5 seconds

d.) Find when the particle is moving left and when the particle is moving right.

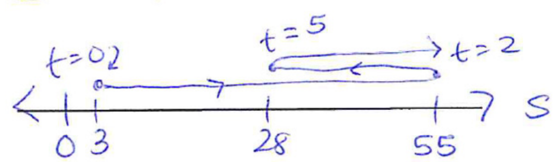
v(t) plug in values to on both sides of 2 and 5



right (0,2) (5,∞)
left (2,5)

e.) Draw a motion diagram for the particle.

$S(2) = 55m$
$S(5) = 28m$



Example: For the displacement function from the last example, $s(t) = 2t^3 - 21t^2 + 60t + 3$, with s in meters and $t \geq 0$ seconds, we found that $v(t) = 6t^2 - 42t + 60$.

a.) Find the average acceleration of the particle from t = 1 second to t = 4 seconds.

$$a_{average} = \frac{v(1) - v(4)}{1-4} = \frac{24-(-12)}{-3} = -12 \text{ m/s}^2$$

or -12 ms^{-2}

b.) Find the instantaneous acceleration of the particle at t = 3 seconds. Explain the meaning of your answer.

$$a(t) = v'(t) = 12t - 42$$

$$a(3) = v'(3) = 12(3) - 42 = -6 \text{ m/s}^2$$

Explain: velocity is decreasing 6 m/s each second at t = 3

Example: For the displacement function from the last example, $s(t) = 2t^3 - 21t^2 + 60t + 3$, with s in meters and $t \geq 0$ seconds, we found that $v(t) = 6t^2 - 42t + 60$ and $a(t) = 12t - 42$.

a.) Find the speed of the particle at t = 3 seconds and determine whether the particle is speeding up or slowing down when t = 3 seconds.

Speed $= |v(t)| = |v(3)| = |-12| = 12$ m/s

Since $a(3) = -6$

The particle is speeding up

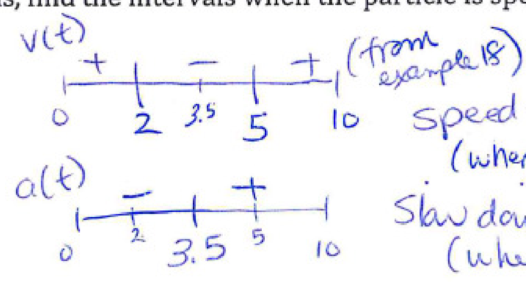
b.) During $0 \leq t \leq 10$ seconds, find the intervals when the particle is speeding up and when it is slowing down.

$a(t) = 12t - 42 = 0$

$t = 3.5$

$v(t) = 0$

$t = 2 + 5$



Speed up $(2, 3.5) \cup (5, 10)$
(where signs are the same)

Slow down $(0, 2) \cup (3.5, 5)$
(where signs are different)